Measuring credit risk in a large banking system: econometric modeling and empirics*

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Abstract

Two new measures for financial systemic risk are computed based on the time-varying conditional and unconditional probability of simultaneous failures of several financial institutions. These risk measures are derived from a multivariate model that allows for skewed and heavy-tailed changes in the market value of financial firms' equity. Our model can be interpreted as a Merton model with correlated Lévy drivers. This model incorporates dynamic volatilities and dependence measures and uses the overall information on the shape of the multivariate distribution. Our correlation estimates are robust against possible outliers and influential observations. For very large cross-sectional dimensions, we propose an approximation based on a conditional Law of Large Numbers to compute extreme joint default probabilities. We apply the model to assess the risk of joint financial firm failure in the European Union during the financial crisis.

Keywords: systemic risk; dynamic equicorrelation model; generalized hyperbolic distribution; Law of Large Numbers.

JEL classification: *G21, C32.*

1 Introduction

We propose a new approach to measure the credit risk in a large system of European financial institutions, based on the time-varying probability of simultaneous failure of multiple financial institutions. Such joint failures are akin to financial crises when the banking sector is in distress. Our measures for joint financial firm failure are based on a dynamic multivariate Generalized Hyperbolic skewed-t (GHST) density that allows for skewed and heavy-tailed changes in the market value of financial firms' equity. The model incorporates dynamic volatilities and failure dependence, while being consistent with expectations about firms' marginal probabilities of failure at each point in time. By applying the new model to the data of European large financial institutions, we show that the model works well even when the cross-sectional dimension is large. Since the model can be treated as a statistical factor model, it can also be used to explore the possible economic variables driving the variation in the default dependence structure.

The systemic risk or the joint default probability of financial institutions has drawn considerable attention since the recent global financial crisis. How to measure the systemic risk and safeguard the financial system during periods of stress has become the key interest of policy makers. There are several commonly used approaches to measure the systemic risk. The Macro stress tests, such as the 2009 SCAP exercise in the U.S. and the 2010 and 2011 CEBS/EBA stress tests in the E.U., are widely used to assess financial stability conditions. However, they are expensive to conduct (both in terms of manpower at supervisory agencies as well as at the involved financial institutions), subject to a wide range of political sensitivities, and as a result not suitable for regular financial sector surveillance at monthly frequency. Model-based Banking Stability Measures (BSM) are considered a valuable alternative to more accurate financial stability assessments. The model proposed in this paper yields two financial stability measures related to the conditional and unconditional default probability of a certain percentage of banks in the system at one point in time. Such banking stability measures are currently widely used in central banks and international policy institutions, see for example ECB (2010).

The construction of useful systemic risk or banking stability measures, however, is not straightforward. First, the risk of a systemic event, such as the simultaneous failure of multiple financial firms, usually involves a high cross-sectional dimension, even if only large and possibly systemically important financial institutions are considered. Extending a copula or multivariate density model beyond, say, five time series is difficult. Second, the failure dependence among financial institutions is time-varying. In particular, the interconnectedness across financial firms appears to be stronger during times of turmoil. For example, fire-sale externalities may connect financial firms through market prices in bad times even in the absence of direct business links, see for example Lorenzoni (2008), Brunnermeier and Pedersen (2009), and Korinek (2011). As a result, taking into account higher correlations during times of stress, in addition to higher marginal risks, is an important feature of financial systemic risk. We overcome the two problems of a high dimension and time-varying parameter values by proceeding in two steps. First, we separate the univariate from the multivariate analysis, as in Engle (2002). Second we impose a parsimonious equicorrelation structure into our dynamic density, similar to the approach taken by Engle and Kelly (2012). The parsimonious structure then ensures that the computations remain tractable. The time variation in volatility and correlation parameters is modeled following the Generalized Autoregressive Score (GAS) framework of Creal, Koopman, and Lucas (2011), and Zhang, Creal, Koopman, and Lucas (2011). In our setting, the scaled score of the local log-likelihood drives the dynamic behavior of the time-varying parameters. As a result, the log-likelihood is available in closed form and can be easily maximized.

Two studies in particular relate to our construction of financial stability measures. In each case, the banking system is seen as a portfolio of financial intermediaries whose multivariate dependence structure is inferred from equity returns. Avesani, Pascual, and Li (2006) assess financial failure in a Gaussian factor model setting. The determination of joint failure probabilities is in part based on the notion of an nth-to-default CDS basket, which can be set up and priced as suggested in Hull and White (2004). Alternatively, Segoviano and Goodhart (2009) propose a non-parametric copula approach. Here, the banking system's multivariate density is recovered by minimizing the distance between the so-called banking system multivariate density and a parametric prior density subject to tail constraints that reflect individual failure probabilities. We regard each of these approaches as polar cases, and attempt to strike a middle ground. The proposed GAS framework in our current paper retains the ability to describe the salient equity data features in terms of skewness, fat tails, and time-varying correlations (which the Gaussian copula fails to do), and in addition retains the ability to fit a cross-sectional dimension larger than fifteen (which the non-parametric approach fails to do due to computational problems). In addition, and for the first time, parameter non-constancy is addressed explicitly in our new modeling setup. The two above approaches are inherently static, and rely on a rolling window approach to capture time variation in parameters. By contrast, we model the parameter dynamics explicitly in a parsimonious way.

The remainder of the paper is structured as follows. Section 2 introduces a framework for simultaneous failures of financial sector firms. The econometric framework is introduced in Section 3 and two new risk measures are proposed in Section 4. Section 5 presents empirical results on the likelihood of joint failures of large financial institutions in the European Union. Section 6 concludes.

2 A framework for simultaneous financial firm failures

The structural approach due to Merton (1974) and Black and Cox (1976) is probably the most widely used approach for the estimation of individual firms' credit risk. In this firm value framework, a firm's underlying asset value evolves stochastically over time, and default is triggered if the firm's asset value falls below a certain default threshold. This threshold is in general determined by a firm's current liability structure. It is straightforward to extend the basic premise of the Merton model to a portfolio credit risk setting. In the case of multiple firms, however, the assumptions regarding the correlation (more generally, dependence) structure between the firm value processes are important for overall risk.

First, consider the simple case of two firms $i = 1, 2$, whose asset values $S_{i,t}$ follow

$$\mathrm{d}S_{i,t} = S_{i,t}(\mu_i \mathrm{d}t + \sigma_i \mathrm{d}W_{i,t}), \tag{1}$$

where $W_{i,t}$ is a standard Brownian Motion, μ_i and σ_i^2 are drift and variance parameters, respectively, and $\mathrm{d}W_{1,t}W_{2,t} = \rho \mathrm{d}t$. The solution to Equation (1) is

$$S_{i,t} = S_{i,0} \exp\left[\left(\mu_i - \sigma_i^2/2\right) t + \sigma_i W_{i,t}\right]. \tag{2}$$

If $\log S_{i,0} = 0$, the log asset values are normally distributed as

$$y_{i,t} = \log S_{i,t} \sim \mathrm{N}\left[\left(\mu_i - \sigma_i^2/2\right) t, \sigma_i^2 t\right]. \tag{3}$$

The use of Brownian Motions and Gaussian distributions has been popular in the literature for modeling asset returns. However, the conditions of Brownian Motions and the log-normal distribution are too restrictive for financial datasets. The asset returns are usually skewed and heavy-tailed, with time-varying (co)variances. The price process does not have a continuous path as the Brownian Motion, but is identified as a semi-martingale with jumps (Cont and Tankov (2004)). To incorporate these empirical features, the Generalized Hyperbolic (GH) Lévy process has gained more attention as a replacement for the Gaussian assumption. The GH distributions are infinitely divisible (Barndorff-Nielsen and Halgreen (1977)) and every member of this family can generate a Lévy process that is a semimartingale. We focus on the GH skewed-t distribution in this paper, which is an asymmetric version of the Student's t distribution. Our analysis can be easily extended to several other GH distributions. Eberlein (2001) provides a useful survey on asset pricing models under the GH Lévy process assumption.

We write the firm values in a Lévy process framework as in Bibby and Sørensen (2001),

$$\mathrm{d}S_{i,t} = \frac{1}{2} v(S_{i,t})[\log(f(S_{i,t})v(S_{i,t}))]' \mathrm{d}t + \sqrt{v(S_{i,t})}\mathrm{d}W_{i,t}, \tag{4}$$

with $v(S_{i,t})$ and $f(S_{i,t})$ two continuously differentiable strictly positive real functions defined on $\mathbb{R}$. Following the arguments in Bibby and Sørensen (2003), we can find suitable functions for a prescribed marginal distribution, for instance a GH skewed-t distribution. The asset value becomes

$$S_{i,t} = S_{0,t} \exp\left(\mathcal{L}_{i,t}\right),$$

where $\mathcal{L}_i$ is a Generalized Hyperbolic Skewed-t Lévy process and the log asset values are Generalized Hyperbolic Skewed-t distributed at discrete time intervals as

$$y_{i,t} = \log S_{i,t} \sim GHST(\tilde{\sigma}^2_{i,t}, \gamma_i, \nu).$$

Compared to the Student's t distribution, the GHST distribution is an asymmetric distribution with γ_i as the skewness parameter. It is flexible enough to capture the most interesting data features with a limited set of parameters. The dynamic version of the GH distribution proposed in Zhang, Creal, Koopman, and Lucas (2011) can accommodate in addition the time-varying covariance matrices. In this paper we adopt the same framework, which is now used to model the correlated defaults in a large portfolio.

In the Merton model and also in our paper, a borrower i defaults at time t if $y_{i,t}$ falls below the firm specific default threshold $y^*_{i,t}$. Therefore, at time t, the firm's marginal probability of default $p_{i,t}$ is given by

$$p_{i,t} = F(y^*_{i,t}), \tag{5}$$

where $F(\cdot)$ is the cumulative distribution function (CDF) of a standard univariate GHST distribution. Similarly, the joint default probability of two borrowers is

$$p_{1\&2,t} = F_\rho\left(y^*_{1,t}, y^*_{2,t}\right), \tag{6}$$

where F_ρ is the bivariate standard GHST distribution function with correlation ρ.

If an estimate of a firm's marginal default probability is available, say from Moody's KMV EDF estimates, then (5) implicitly defines the corresponding threshold value $y^*_{i,t}$. With these thresholds, we are able to determine a distress region for the multivariate distribution. A firm defaults at time t when its asset value $y_{i,t}$ fall into the region $(-\infty, y^*_{i,t})$. In this paper, we adopt EDF estimates as the estimated probability of default.

3 The model

3.1 The Dynamic GH skewed-t model

The risk measure we propose is the joint default probability for a large portfolio of N banks. In the multivariate case, the joint default probability can be inferred from the market by considering the interrelationship of equity returns. We assume the equity returns $y_t = (y_{1,t}, \cdots, y_{N,t})'$ follow a multivariate dynamic Generalized Hyperbolic skewed-t (GHST) distribution. The GHST distribution can be obtained as a normal mean-variance mixture

$$y_t = (\varsigma_t - \frac{\nu}{\nu-2})\tilde{L}_t\gamma + \sqrt{\varsigma_t}\tilde{L}_t\epsilon_t, \tag{7}$$

with a scalar random variable $\varsigma_t \sim \text{InverseGamma}(\nu/2, \nu/2)$ where ς_t is independent of ϵ_t, and N-dimensional $\epsilon_t \sim \mathrm{N}(0, \mathrm{I}_N)$, and $\tilde{L}_t$ is an $N \times N$ loading matrix which defines the individual exposures to the common risk factor ϵ_t. The mixing structure introduces non-trivial clustering in the tails compared to the situation with only a Gaussian factor ϵ_t. The GHST density of y_t is given by

$$\begin{aligned} p(y_t; \tilde{\Sigma}_t, \gamma, \nu) &= \frac{\nu^{\frac{\nu}{2}} 2^{1-\frac{\nu+N}{2}}}{\Gamma(\frac{\nu}{2})\pi^{\frac{N}{2}}|\tilde{\Sigma}_t|^{\frac{1}{2}}} \cdot \frac{K_{\frac{\nu+N}{2}}\left(\sqrt{d(y_t)\cdot(\gamma'\gamma)}\right) e^{\gamma'\tilde{L}_t^{-1}(y_t-\tilde{\mu}_t)}}{d(y_t)^{\frac{\nu+N}{4}} \cdot (\gamma'\gamma)^{-\frac{\nu+N}{4}}}, && (8) \\ d(y_t) &= \nu + (y_t - \tilde{\mu}_t)'\tilde{\Sigma}_t^{-1}(y_t - \tilde{\mu}_t), && (9) \\ \tilde{\mu}_t &= -\frac{\nu}{\nu-2}\tilde{L}_t\gamma, && (10) \end{aligned}$$

where $K_a(b)$ is the modified Bessel function of the second kind, $\tilde{\Sigma}_t = \tilde{L}_t\tilde{L}_t'$ is the scale matrix, see Bibby and Sørensen (2003).

$$\begin{aligned} \tilde{L}_t &= L_t T, & (11) \\ (T'T)^{-1} &= \frac{\nu}{\nu-2}\mathrm{I} + \frac{2\nu^2}{(\nu-2)^2(\nu-4)}\gamma\gamma', & (12) \end{aligned}$$

The matrix L_t characterizes the time-varying covariance matrix $\Sigma_t = L_t L_t'$. We consider the time-varying covariance matrix of y_t as

$$\Sigma_t = L_t L_t' = D_t R_t D_t, \tag{13}$$

where D_t is a diagonal matrix holding the volatilities of $y_{i,t}$ and R_t is the correlation matrix of equity returns y_t. The marginal distribution for a multivariate Generalized Hyperbolic skewed-t distribution is a univariate Generalized Hyperbolic skewed-t distribution. The skewness variables can be different in each marginal.

We assume the dynamic covariance matrix Σ_t depends on the unobserved factor f_t, where f_t follows the Generalized Autoregressive Score process as specified in Creal, Koopman and Lucas (2011, 2012) and Zhang, Creal, Koopman, and Lucas (2011).

$$f_{t+1} = \omega + \sum_{i=0}^{p-1} A_i s_{t-i} + \sum_{j=0}^{q-1} B_j f_{t-j}, \tag{14}$$

$$s_t = S_t \nabla_t, \tag{15}$$

$$\nabla_t = \partial \ln p_{GH}(y_t|\mathcal{F}_{t-1}; f_t, \theta)/\partial f_t, \tag{16}$$

ω is a vector of fixed intercepts, and A_i and B_j are fixed parameter matrices. In order to obtain our result below, we define

$$\mathrm{vec}(L) = \mathcal{D}_N^0 \mathrm{vech}(L) \tag{17}$$

for a $N \times N$ lower triangular matrix L,

$$\text{vech}(S) = \mathcal{B}_N \text{vec}(S) \tag{18}$$

for a symmetric matrix S, and the commutation matrix $\mathcal{C}_N$ for an $N \times N$ matrix X as

$$\text{vec}(X) = \mathcal{C}_N \text{vec}(X'). \tag{19}$$

Result 1. *If y_t follows a GHST distribution $p(y_t; \tilde{\Sigma}_t, \gamma, \nu)$, where the time-varying covariance matrix is driven by the GAS model* (14)-(16)*. The dynamic score is*

$$\begin{aligned}
\nabla_t &= \Psi_t' H_t' \text{vec}\left(w_t \cdot y_t y_t' - \tilde{\Sigma}_t - \left(1 - \frac{\nu}{\nu-2} w_t\right) \tilde{L}_t \gamma y_t' \right), &(20)\\
w_t &= \frac{\nu + N}{2d(y_t)} - \frac{k'_{\frac{\nu+N}{2}}\left(\sqrt{d(y_t)\cdot(\gamma'\gamma)}\right)}{2\sqrt{d(y_t)/(\gamma'\gamma)}}, &(21)\\
\Psi_t &= \frac{\partial \text{vech}(\Sigma_t)'}{\partial f_t}, &(22)\\
H_t &= (\tilde{\Sigma}_t^{-1} \otimes \tilde{\Sigma}_t^{-1})(\tilde{L}_t \otimes \mathrm{I}_N)(T' \otimes \mathrm{I}_N)\mathcal{D}_N^0 (\mathcal{B}_N(\mathrm{I}_{N^2} + \mathcal{C}_N)(L_t \otimes \mathrm{I}_N)\mathcal{D}_N^0)^{-1}, &(23)
\end{aligned}$$

where we define $k_{\frac{\nu+N}{2}}(\cdot) = \ln K_{\frac{\nu+N}{2}}(\cdot)$ with first derivative $k'_{\frac{\nu+N}{2}}(\cdot)$. The matrices Ψ_t and H_t are time-varying, parameterization specific, and depend on f_t, but not on the data.

The dynamics driven by the score ∇_t can be seen as a local improvement of the likelihood to the new data observed at time t, and $\mathcal{S}_t$ is a scaling matrix for the score ∇_t. Typical choices for the scaling matrix $\mathcal{S}_t$ are the unit matrix or inverse (powers) of the Fisher information matrix $\mathcal{I}_{t-1}$, where

$$\mathcal{I}_{t-1} = \mathrm{E}\left[\nabla_t \nabla_t' \middle| y_{t-1}, y_{t-2}, \ldots\right].$$

For example, $\mathcal{S}_t = \mathcal{I}_{t-1}^{-1}$ accounts for the curvature in the score ∇_t. With the choice of scaling matrix as the inverse Fisher information matrix, the GAS step s_t can be seen as a Gauss-Newton improvement step of the local fit of the model. As the Fisher information matrix for the GH distribution has no analytical expression, we instead use the inverse Fisher information matrix from the Student's t in our current paper. Zhang, Creal, Koopman, and

Lucas (2011) demonstrate that this results in a stable model that outperforms alternative models if the data are fat-tailed and skewed. We obtain

$$\mathcal{S}_t = \left\{ \Psi_t'(\mathrm{I} \otimes \tilde{L}_t^{-1})'[gG - \mathrm{vec}(\mathrm{I})\mathrm{vec}(\mathrm{I})'](\mathrm{I} \otimes \tilde{L}_t^{-1})\Psi_t \right\}^{-1}, \tag{24}$$

where $g = (\nu + N)/(\nu + 2 + N)$, and $G = \mathrm{E}[x_t x_t' \otimes x_t x_t']$ for $x_t \sim \mathrm{N}(0, \mathrm{I}_N)$.

3.2 Estimation and restrictions

Zhang, Creal, Koopman, and Lucas (2011) show that the GAS dynamic structure has superior performance under skewed and fat-tailed distributions. However, evaluating the full covariance matrix in the full likelihood is cumbersome computationally if the dimension of the data is large. Therefore, we separate the estimation of the covariance matrix into volatility estimation and correlation estimation procedures. The algorithm works in two steps.

1. Estimate the log-volatility $\log(\sigma_{it})$ for each series with a univariate dynamic GHST model. The skewness parameter is estimated for each series separately, but the kurtosis parameter is fixed at 5. The motivation is to ensure that the marginal GHST distributions are internally consistent with the multivariate GHST distribution. The data at time t is standardized by the volatility σ_{it}. The standardized data is tested for serial correlation using the F-test suggested in Engle (2002).

2. Estimate the correlation matrix R_t of the standardized returns using the volatilities from the first step. The correlation matrix is driven by the factor f_t from the multivariate dynamic GHST model. Again the kurtosis parameter is set ex ante as $\nu = 5$ and the skewness parameters are equal to those from the univariate distributions obtained from the first step. We need a parametrization as in Engle (2002) or Zhang, Creal, Koopman, and Lucas (2011) to ensure that R_t actually is a correlation matrix.

In the univariate and the multivariate GH skewed-t model, we fix the degrees of freedom parameter for all the marginal distributions at five. We can also estimate a GHST distribution in order to obtain a sensible degree of freedom. Interestingly when estimate static GHST model in a exploratory analysis, we find five a reasonable parameter value that ensures the distribution captures the tail behavior of the data.

The idea behind the algorithm is simple. We first use the dynamic GHST model as a filter for the volatility in the equity returns for each of the series. The standardized equity returns are then used in a multivariate dynamic GHST model model, where the covariance matrix is the correlation matrix. It is similar to the two-step procedure or the composite likelihood method in Engle (2002), Hu (2005), and other studies that are based on a multivariate GARCH framework.

If we want to work with a large dimensional dataset, we still need to impose some further restrictions to confront the computational difficulties. One difficulty arises from estimating the unconditional mean ω in Equation (14). In a dataset of N time series, we have to estimate $N(N-1)/2$ coefficient for the unconditional mean of factors ω. In order to reduce the computational difficulty, we estimate the unconditional mean of the factors $\bar{f} \in \mathbb{R}^{N(N-1)/2}$ separately and estimate a scalar ω in the equation (14),

$$f_{t+1} = \omega \bar{f} + \sum_{i=0}^{p-1} A_i s_{t-i} + \sum_{j=0}^{q-1} B_j f_{t-j}; \tag{25}$$

The ω is now defined as the levels of correlation coefficients proportional to the unconditional mean of our factors. We choose A and B scalar parameters as in the DCC model. This reduces the total number of parameters in GAS model to three only, irrespective of the data's cross-sectional dimension. In practice, we can also fix ω at one, because the parameter estimate is usually close to one. It is sometimes called "correlation targeting" in the literature.

One of the attractive features of the GAS model is the possibility to introduce a latent factor structure to describe the time variation in the dynamic parameters we are interested in. We could impose the restriction that several time-varying parameters are driven by common factors. This is extremely useful to process high-dimensional data from a large system. In the next section, we introduce the block GAS-Equicorrelation model and the

GAS-Equicorrelation model as two examples of such a framework.

3.3 The Block GAS-Equicorrelation model

With the two-step estimation procedure, the task of maximizing the multivariate GHST likelihood in a large system becomes more feasible. The computational burden is largely reduced due to the separation of the likelihood for volatilities and correlations. Still, this method is cumbersome if the data dimension becomes high, for instance around 100. The advantage of the factor structure in the GAS framework (14) underlying the dynamic correlation matrix makes it possible to address this problem by using common factors. We assume the factor dimension to be smaller than the number of correlations. This defines a multi-factor structure underlying the dynamic correlation model. In the literature, we call correlation matrices with such a structure a block dynamic equicorrelation matrix. Assume that N firms fall into m different groups according to their exposure to a common systemic risk factor. Firms have equicorrelation ρ_i^2 within each group and $\rho_i \cdot \rho_j$ between groups i and j. So we have $N = n_1 + n_2 + \cdots + n_m$ random variables that follow a GH distribution with a correlation matrix that has a block equicorrelation structure, where n_i denotes the number of firms in group i. The correlation matrix at time t is given by

$$R_t = \begin{bmatrix} (1-\rho_{1,t}^2)\mathrm{I}_{n_1} & \dots & \dots & 0 \\ 0 & (1-\rho_{2,t}^2)\mathrm{I}_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1-\rho_{m,t}^2)\mathrm{I}_{n_m} \end{bmatrix} + \begin{pmatrix} \rho_{1,t}\ell_1 \\ \rho_{2,t}\ell_2 \\ \vdots \\ \rho_{m,t}\ell_m \end{pmatrix} \cdot \begin{pmatrix} \rho_{1,t}\ell_1' & \rho_{2,t}\ell_2' & \dots & \rho_{m,t}\ell_m' \end{pmatrix}, \tag{26}$$

where $\ell_i \in \mathbb{R}^{n_i \times 1}$ is a column vector of ones and $|\rho_{i,t}| < 1$ to ensure the positive-definiteness of R_t. The matrix L_t and the inverse of L_t can be calculated explicitly by assuming

$$L_t = \begin{bmatrix} a_{1,t}\mathrm{I}_{n_1} & \dots & \dots & 0 \\ 0 & a_{2,t}\mathrm{I}_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{m,t}\mathrm{I}_{n_m} \end{bmatrix} + \begin{bmatrix} b_{11,t}J_{11} & b_{12,t}J_{12} & \dots & b_{1m,t}J_{1m} \\ b_{12,t}J_{21} & b_{22,t}J_{22} & \dots & b_{2m,t}J_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1m,t}J_{m1} & b_{2m,t}J_{m2} & \dots & b_{mm,t}J_{mm} \end{bmatrix}, \tag{27}$$

where $J_{ij} \in \mathbb{R}^{n_i \times n_j}$ is a matrix of ones $J_{ij} = \ell_i \ell_j'$. We solve for all the parameters in the equation $R_t = L_t L_t'$, where L_t is symmetric. The block equicorrelation model allows us to obtain analytical solutions for the determinant of R_t. As a result of the Matrix Determinant Lemma (see Harville (2008)), the determinant of the matrix R_t is

$$\begin{aligned}\det(R_t) &= \det(\Xi_t + u_t u_t') = (1 + u_t' \Xi_t^{-1} u_t) \det(\Xi_t) \\ &= \left[1 + \frac{n_1 \rho_{1,t}^2}{1-\rho_{1,t}^2} + \cdots + \frac{n_m \rho_{m,t}^2}{1-\rho_{m,t}^2}\right] (1-\rho_{1,t}^2)^{n_1} \cdots (1-\rho_{m,t}^2)^{n_m},\end{aligned}$$

with Ξ_t the diagonal matrix in the first term on the righthand side of (26) and u_t the vector in the second term, such that $R_t = \Xi_t + u_t u_t'$. The determinant of matrix L_t is easy to find as the square root of this value. The analytic expressions facilitate the computation of the likelihood and GAS steps in high dimensions. The time-varying correlation coefficients $\rho_{1,t}, \cdots, \rho_{m,t}$ are driven by the GAS factors from a GH skewed-t distribution. We can derive the GAS model with these restrictions.

Result 2. *If y_t follows a GH skewed-t distribution and the time-varying correlation matrix R_t has a block equicorrelation structure, the dynamic score follows Equation* (20) *and the matrix H_t stays the same as Equation* (23). *We denote the time-varying parameters in R_t as $\Phi_t = (\rho_{1,t}, \cdots, \rho_{m,t})' = f_t$. The major difference is $\frac{\partial \text{vec}(R_t)'}{\partial f_{i,t}}$ as part of $\frac{\partial \text{vec}(\Sigma_t)'}{\partial f_t}$ in Ψ_t,*

$$\frac{\partial \text{vec}(R_t)'}{\partial f_{i,t}} = -2\rho_{i,t} \cdot \text{vec} \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \mathrm{I}_{n_i} & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} + \left[\begin{pmatrix} \rho_{1,t}\ell_1 \\ \rho_{2,t}\ell_2 \\ \vdots \\ \rho_{m,t}\ell_m \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \vdots \\ \ell_{n_i} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ \ell_{n_i} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \rho_{1,t}\ell_1 \\ \rho_{2,t}\ell_2 \\ \vdots \\ \rho_{m,t}\ell_m \end{pmatrix} \right] \tag{28}$$

The simplest case of the block GAS-Equicorrelation model is if we only have one block, which we call the GAS-Equicorrelation model. Following Engle and Kelly (2012), we then assume the correlation matrix R_t with the equicorrelation structure:

$$R_t = (1-\rho_t)\mathrm{I} + \rho_t \ell \ell', \tag{29}$$

where $\rho_t \in (\frac{-1}{N-1}, 1)$. Under such an assumption, the dynamic score equation stays the same as (20), but the matrix computations are simplified.

Result 3. *If we assume one equicorrelation structure for the correlation matrix, the GAS model works as in the equations in Section 3.1. The only difference is that Ψ_t simplifies to:*

$$\Psi_t = \left(\frac{da_t}{d\rho_t} \mathrm{cvec}(\mathrm{I}_N) + (\frac{da_t}{d\rho_t} d + \frac{db_t}{d\rho_t} c + \frac{db_t}{d\rho_t} N d) \ell_{N^2} \right) \vartheta'(f_t), \tag{30}$$

$$\frac{da_t}{d\rho_t} = -\frac{1}{2\sqrt{1-\rho_t}}, \tag{31}$$

$$\frac{db_t}{d\rho_t} = \frac{1}{2N} \left(\frac{N-1}{\sqrt{1-\rho_t+N\rho_t}} + \frac{1}{\sqrt{1-\rho_t}} \right), \tag{32}$$

where the scalar $c = \frac{1}{\sqrt{\mu_\varsigma}}$, $d = \frac{\sqrt{\mu_\varsigma} - \sqrt{\sigma_\varsigma^2 \gamma' \gamma + \mu_\varsigma}}{\sqrt{\mu_\varsigma}}$, $\mu_\varsigma = \frac{\nu}{\nu-2}$, *and* $\sigma_\varsigma^2 = \frac{2\nu^2}{(\nu-2)^2(\nu-4)}$.

The GAS-Equicorrelation model may seem too restrictive at first. In our application, however, the data we are dealing with are European financial institutions that have strong economic and financial links and the equicorrelation captures our salient parameter of interest: the systemic dynamic correlation in the entire system of banks considered. We compare the equicorrelation model with the full GAS model in Section 5.1 for a small system where we can still estimate both models. For the large system with more than 70 institutions, we only consider the GAS-Equicorrelation version of the model.

4 The risk measures in a large system

There are multiple ways to construct a financial sector stability measure. For example, a higher probability of at least a certain number of firms failing over the next year is a natural measure of systemic risk. Such a measure is for example constructed and tracked in the European Central Bank's biannual Financial Stability Report, see for example ECB (2010). Here we use the same definition of a systemic risk measure. After estimating the conditional covariance matrix through the dynamic-GH model, the time-varying correlation and volatility mechanism are used to calculate the probability of failure of European financial firms. With this estimated multivariate density, we can thus produce a systemic risk measure. In this section, we calculate this measurement either by simulation or by analytic approximations. The latter are particularly useful for large cross-sectional dimensions.

The straightforward approach is based on simulations of equity returns. As discussed in Section 2, a firm default may happen if the equity return is too negative compared to pre-specified default threshold. In the multivariate distribution, these thresholds define a distress region. We can generate simulations and compute tail probabilities by counting the number of realizations in this pre-determined distress region. In this paper, we simulate from the estimated dynamic multivariate GHST distribution. The distress region is determined by the default thresholds transformed from Moody's EDF estimates. This simulation based method is general enough for all different distributions and model specifications.

When the dimension of the dataset becomes too large, the simulation based risk measurements become inefficient. We need a large number of simulations. Interestingly, we are able to explore the advantage of the equicorrelation structure for the simplified correlation matrix. This is the alternative approach to produce the systemic risk in a large system. We consider the system of banks as homogenous portfolio of equities.[1] We can use a Law of Large Number (LLN) result in the context of credit risk as in Lucas et al. (2001). We define the Systemic Risk indicator as the probability that a certain number of banks default in the same timespan. The number of defaults at time t is

$$c_{N,t} = \frac{1}{N}\sum_{i=1}^{N} 1\{y_{i,t} < y^*_{i,t}|\kappa_t, \varsigma_t\}. \tag{33}$$

Given that the $1\{y_{i,t} < y^*_{i,t}\}$s are conditionally independent, the Law of Large Numbers tells us if $N \to +\infty$,

$$\begin{aligned} c_{N,t} &\approx \frac{1}{N}\sum_{i=1}^{N} \mathrm{E}(1\{y_{i,t} < y^*_{i,t}|\kappa_t, \varsigma_t\}) && (34) \\ &= \frac{1}{N}\sum_{i=1}^{N} \mathrm{P}[y_{i,t} < y^*_{i,t}|\kappa_t, \varsigma_t]. && (35) \end{aligned}$$

[1] The homogeneity assumption is only used for exposition. Different γ_i and ρ_i in the block equicorrelation structure can easily be allowed for.

If the returns are GHST distributed and have a block equicorrelation structure as equation (26), we can model the banks' market values as:

$$
\begin{aligned}
y_t &= (\varsigma_t - \mu_\varsigma)\gamma + \sqrt{\varsigma_t} z_t, && (36)\\
z_t &= \eta_t \kappa_t + \Lambda_t \epsilon_t, && (37)
\end{aligned}
$$

where $\kappa_t \sim \mathrm{N}(0,1)$ and $\epsilon_t \sim \mathrm{N}(0, \mathrm{I}_N)$, η_t is a vector of parameters $(\eta_{1,t}, \cdots, \eta_{N,t})'$, and Λ_t is an $N \times N$ diagonal matrix with $(\lambda_{1,t}\ell_1', \cdots, \lambda_{m,t}\ell_m')$ on the diagonal. We are interested in finding the values of η_t and Λ_t such that $\mathrm{Var}(z_t) = R_t$. We know

$$
\begin{aligned}
\mathrm{Var}(y_t) &= \Xi_t + u_t u_t' \\
&= \mu_\varsigma \Lambda_t^2 + \mu_\varsigma \eta_t \eta_t' + \sigma_\varsigma^2 \gamma\gamma'. && (38)
\end{aligned}
$$

So the parameters η_t and Λ_t should satisfy the following equations,

$$
\begin{aligned}
\lambda_{i,t} &= \sqrt{\frac{1-\rho_{i,t}^2}{\mu_\varsigma}}, \text{ for } i = 1, \cdots, m, && (39)\\
\mu_\varsigma \eta_t \eta_t' &= u_t u_t' - \sigma_\varsigma^2 \gamma\gamma'. && (40)
\end{aligned}
$$

This is a two-factor model with a common Gaussian factor κ_t and a mixing factor ς_t. The stability measure in this setting is given by

$$
p_t = \mathrm{P}(C_{N,t} > c_{p,t}), \tag{41}
$$

where we can compute the measure conditional on the latent factors κ_t and ς_t,

$$
\begin{aligned}
c_{p,t} &= \frac{1}{N}\sum_{i=1}^{N} \mathrm{P}[y_{i,t} < y_{i,t}^* | \kappa_t, \varsigma_t], && (42)\\
\mathrm{P}[y_{i,t} < y_{i,t}^* | \kappa_t, \varsigma_t] &= \Phi\left(\left.\frac{(y_{i,t}^* + \mu_\varsigma\gamma_i - \varsigma_t\gamma_i)/\sqrt{\varsigma_t} - \eta_{i,t}\kappa_t}{\lambda_{i,t}}\right| \kappa_t, \varsigma_t\right). && (43)
\end{aligned}
$$

The risk measure is related to the number of defaults as a proportion in the portfolio. Using equation (42), we rewrite the threshold common factor $\kappa_t = \kappa_t^*(c_{p,t}, \varsigma)$ as a function of the default proportion $c_{p,t}$ and the mixing variable ς_t. We are able to compute the joint default probability numerically as

$$p_t = \mathrm{P}(C_{N,t} > c_{p,t}) = \int \mathrm{P}(\kappa_t < \kappa_t^*(c_{p,t}, \varsigma_t))p(\varsigma_t)\mathrm{d}\varsigma_t. \tag{44}$$

Similarly, we can compute the probability of certain proportion $c_{p,t}^{-i}$ of the system excluding bank i defaulting conditional on the event that bank i fails.

$$\begin{aligned}
\mathrm{P}(C_{N-1,t} > c_{p,t}^{-i}|y_{i,t} < y_{i,t}^*) &= \frac{\mathrm{P}(C_{N-1,t} > c_{p,t}^{-i}, y_{i,t} < y_{i,t}^*)}{\mathrm{P}(y_{i,t} < y_{i,t}^*)} \\
&= \frac{\int \Phi_2(\frac{z_{i,t}^*\sqrt{\mu_\varsigma}}{\sqrt{1-\sigma_\varsigma^2\gamma'\gamma}}, \kappa_t^*(c_{1,t}^i, \varsigma_t), \eta_{i,t})p(\varsigma_t)\mathrm{d}\varsigma_t}{\int \mathrm{P}(\kappa_t < \kappa_t^*(c_{1,t}^i, \varsigma_t))p(\varsigma_t)\mathrm{d}\varsigma_t},
\end{aligned} \tag{45}$$

where

$$z_{i,t}^* = \frac{y_{i,t}^* - (\varsigma_t - \mu_\varsigma)\gamma_i}{\sqrt{\varsigma_t}} \tag{46}$$

from Equation (36), $\Phi_2(\cdot, \cdot, \eta_{i,t})$ is the bivariate normal CDF with correlation $\eta_{i,t}$, and $\kappa_t^*(c_{1,t}^i, \varsigma_t)$ denotes the corresponding threshold common factor when bank i's equity return fall below the threshold y_i^*. This conditional probability is close to the Multivariate extreme spillovers indicator of Hartmann, Straetmans, and de Vries (2005).

We define the average of this conditional default probability over N financial firms as the Systemic Risk Measure (SRM), as it measures the possibility that an individual credit event increases the level of systemic risk. We apply the two measurements proposed here in the empirical section.

5 Empirical application

In this section, we compute the banking stability measure in the European Union. We observe 73 major financial groups with complex interactions. The data contain monthly observations of equity prices and estimated EDFs for all 73 financial institutions. Our whole sample covers the period January 1992 to June 2010, but with missing observations of several names in the beginning of the sample. Dealing with missing values in our model's setting is straightforward. Both the likelihood and the score steps in the dynamic GHST model adapt automatically if data are not observed at particular times and there are no sample selection issues.

The analysis in this section consists of two parts. To compare the dynamic GHST model with the block GAS Equicorrelation models, we choose a subsample consisting of ten European banks. The full multivariate model from Section 3.1 is estimated with a time-varying covariance matrix. We also show the estimation results for models in Section 3.3. These results are presented in Section 5.1. Second, we impose the GAS Equicorrelation structure in the dynamic GHST model for the whole sample of 73 financial institutions. The conditional Law of Large Numbers approximation is implemented to compute the Banking Stability Measure and the Systemic Risk Measure. Section 5.2 includes the results for this analysis.

5.1 The system of major European banks

In our first analysis, we select a geographically diversified sub-sample of 10 banks in the Euro Area: Bank of Ireland, BBVA, Santander, BNP Paribas, Commerzbank, Deutsche Bank, Societe Generale, ING, UniCredito, National Bank of Greece. To estimate the time-varying correlations and volatilities, we use monthly log returns from January 1994 to June 2010 from Bloomberg. The dataset contains 198 observations for each series. The EDF data used to compute the distress thresholds are provided by Moody's KMV. From the descriptive statistics in Table 1 we see that all equity returns are skewed and fat-tailed. Commerzbank and ING Group stand out with a pronounced skewness of -1.10 and -1.64, and a kurtosis of 8.33 and 6.99, respectively. However, the Bank of Ireland has a large kurtosis of 16.053. We model the equity returns from all 10 banks with our skewed and heavy-tailed dynamic GH skewed-t model.

We first estimate the full correlation matrix with forty-five pair-wised dynamic corre-

Table 1: Sample Descriptive Statistics.

The descriptive statistics for the monthly equity returns between January 2000 and June 2010. The sample mean values are all very close to zeros. The standard deviations, minimum and maximum values are multiplied by 100 respectively in the table. All skewness and excess kurtosis are significantly different from 0.

	Mean	Std.Dev.	Skewness	Kurtosis	Minimum	Maximum
Bank of Ireland	0.000	1.309	-0.594	16.053	-113.917	106.153
BBVA	0.000	0.710	-0.512	3.220	-38.894	37.003
Santander	0.000	0.720	-0.725	3.758	-40.720	37.609
BNP Paribas	0.000	0.675	-0.502	3.261	-34.001	32.959
Commerzbank	0.000	0.940	-1.101	5.474	-67.779	45.536
Deutsche Bank	0.000	0.760	-0.421	3.906	-46.588	45.444
Societe Generale	0.000	0.777	-0.968	4.110	-53.679	29.201
ING	0.000	0.896	-1.647	8.939	-73.367	45.187
UniCredito	0.000	0.752	-0.048	3.282	-44.318	36.017
National Bank of Greece	0.000	0.938	0.336	2.324	-48.178	53.652

lations driven by the scaled autoregressive scores. The dynamics of these correlations are different over time, but they share some commonality. For instance, all correlations go up during the financial crisis, especially after the failure of Lehman Brothers in September 2008. Figure 2 depicts the correlation series of the other nine banks with the Bank of Ireland, which received recapitalization and a bail-out from the Irish government and ECB in 2009 and 2010. The correlations show a significant drop around the year 2001 and rise during the financial crisis 2008 and onwards.

The estimated volatility series are plotted in separate panels in Figure 1. The volatility estimates are obtained via estimation of the GH skewed-t distribution for each individual time series. All parameters in the volatility models are significant at the 5% significant level, as shown in Table 2. From the graph, we see three highly volatile periods corresponding to either financial crises or global economic recessions. The most recent period with clearly high volatility begins in Sept. 2008, when the failure of Lehman Brothers brought down the stock prices of all banks. But the magnitude of this increase differs from one institution to the other. The most volatile time series is the Bank of Ireland's equity return. In the midst of the Global Financial Crisis, the Irish Banking Crisis hits this largest Irish bank even harder. The Bank of Ireland was recapitalized by the Irish Government in February 2009 and further bailed-out by the ECB and IMF in 2010. The idiosyncratic shock to the

Bank of Ireland, on top of the common shock from the Lehman Brother's bankruptcy, drives up its volatility even higher.

We filter the equity returns with the estimated volatilities and apply a multivariate GH skewed-t model in the second step. The time-varying correlation matrices are assumed to follow the GAS model in Equations (14) and (16). We implement four dynamic GHST models imposing different parameterizations on the dynamic correlation matrix.

As a comparison, we estimate the dynamic GH skewed-t model with the GAS-Equicorrelation model (Equations (29)-(32)), and the two-Block GAS-Equicorrelation model (Equations (26)-(28)) on the same sample. The banks are separated into two groups. The first group contains the Bank of Ireland, BBVA, Santander, UniCredito and the National Bank of Greece. The second group includes the rest banks. The correlation estimates are plotted in the bottom panels in Figure 2. As benchmarks, we also include the average correlation from the Rolling Window (RW) method with the window size set to 12 months.

If we compare the Equicorrelation model outputs and the average correlation from the GAS model and RW method, the dynamic equicorrelation appears to be an average of the pairwise correlations. The flexible GAS-GHST model allows for more heterogenous dynamics on the pair-wise correlation coefficients. But we also see that the equicorrelation model picks up the most salient comovements in the data, such as the drop of correlation in 2001 and the increase after 2008 due to the financial crisis. In the model estimates from the two-block GAS-Equicorrelation matrix, we see that the three correlation estimates exhibit similar time-varying patterns as the equicorrelation dynamics. But we start to see differences in particular periods, for instance around the year 2008. It seems that the correlation of banks in the first group is higher in the crisis period. We provide the parameter estimates and log-likelihood values from the dynamic correlation models in Table 2.

With the estimated GH skewed-t distributions, either with the full model or with the equicorrelations and block equicorrelations, we can compute the Banking Stability Measure (BSM) and Systemic Risk Measure (SRM) given the default thresholds from inverting the GH skewed-t CDF at the observed EDF levels. The banking stability measure is defined as the joint probability of three or more banks defaulting. The Systemic Risk Measure is

Table 2: The Estimation Results: Part I.

The parameter estimated in our GAS-GHST models for ten banks' equity returns. We use univariate GAS-GHST models for the marginal volatility. With the filtered returns, we estimate three dynamic correlation models: the GAS Equicorrelation model, the Block GAS Equicorrelation model, and the GAS model with full correlation structure. All parameters are significant at the 5% level.

	Dynamic Volatility				
	A	B	ω	γ	Log-lik
Bank of Ireland	0.201	0.964	0.093	-0.206	-725.655
	(0.003)	(0.002)	(0.005)	(0.004)	
BBVA	0.154	0.902	0.220	-0.145	-701.432
	(0.003)	(0.004)	(0.010)	(0.004)	
Santander	0.196	0.884	0.256	-0.163	-696.317
	(0.004)	(0.005)	(0.011)	(0.004)	
BNP Paribas	0.212	0.866	0.295	-0.152	-691.252
	(0.005)	(0.006)	(0.014)	(0.005)	
Commerzbank	0.168	0.929	0.175	-0.167	-738.160
	(0.003)	(0.003)	(0.006)	(0.004)	
Deutsche Bank	0.168	0.910	0.211	-0.105	-715.436
	(0.003)	(0.004)	(0.010)	(0.005)	
Societe Generale	0.196	0.918	0.189	-0.134	-711.646
	(0.003)	(0.003)	(0.008)	(0.005)	
ING	0.167	0.915	0.200	-0.224	-719.552
	(0.003)	(0.003)	(0.007)	(0.004)	
UniCredito	0.126	0.969	0.071	-0.064	-708.966
	(0.003)	(0.002)	(0.004)	(0.005)	
National Bank of Greece	0.141	0.927	0.188	-0.060	-768.016
	(0.003)	(0.003)	(0.008)	(0.005)	

	Dynamic Correlation							
	A	B	ω_1	ω_2	γ	Log-lik	AIC	BIC
GAS EquiCorr (1)	0.116	0.915	0.205		-0.071	-2050.956	4111.91	4128.35
	(0.054)	(0.070)	(0.210)		(0.024)			
GAS EquiCorr (2)	0.070	0.907	0.931	1.417		-2052.116	4114.23	4130.67
	(0.029)	(0.057)	(0.161)	(0.148)				
GAS Model	0.027	0.717	1.007			-1952.200	3914.40	3930.84
	(0.009)	(0.099)	(0.009)					

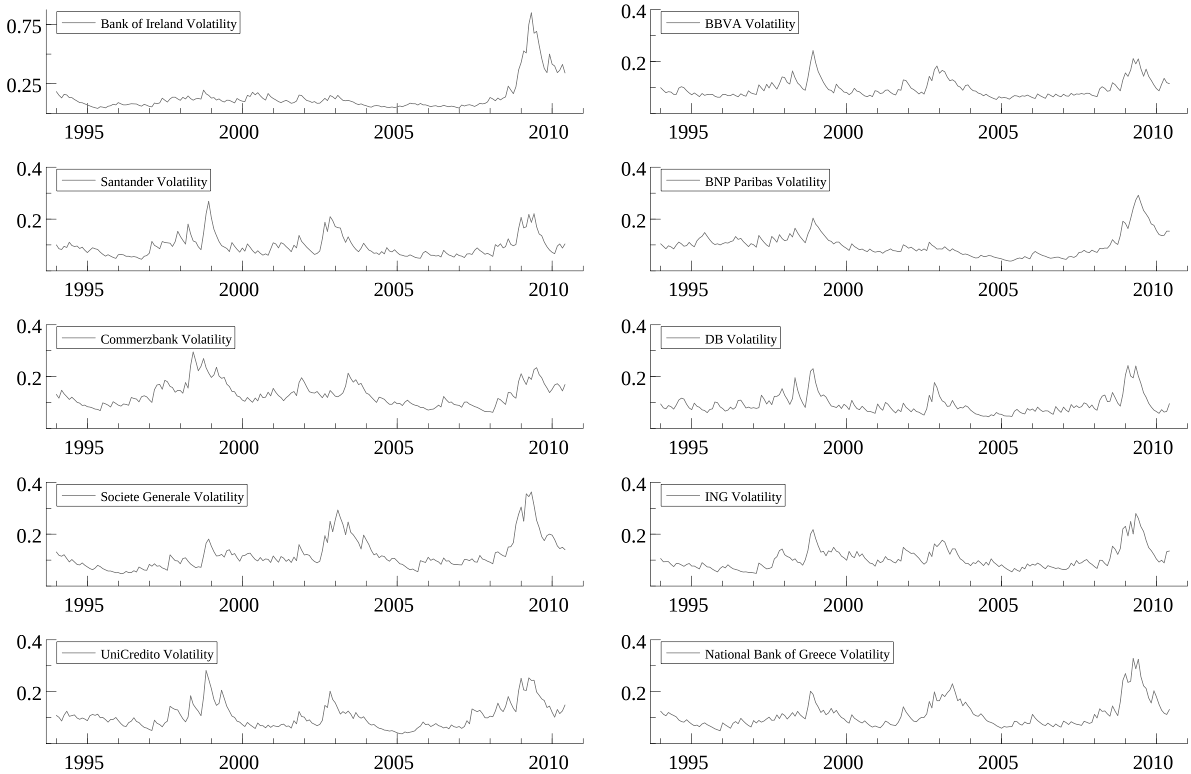

Figure 1: Volatility estimations for the banks' equities

The volatility estimates from the Dynamic GH Skewed-t for all the banks' stock return data. (BBVA stands for BBV.Argentaria and DB refers to Deutsche Bank.)

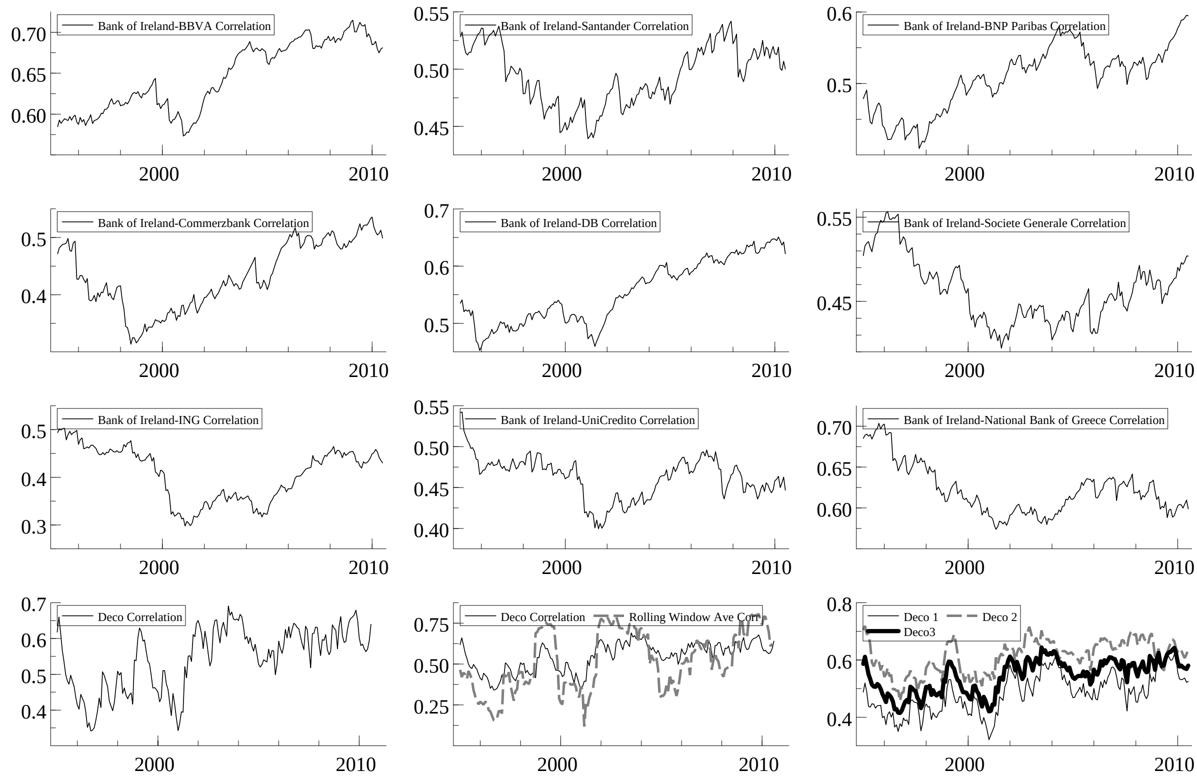

Figure 2: Correlation estimations between the other banks and the Bank of Ireland

The correlation estimates from Dynamic GH Skewed-t model with banks' stock returns. We selected the correlations of the Bank of Ireland's with other banks in our sample. The last two panels are from a one-factor and two-factor equicorrelation model in the skewed-t distribution.

constructed with the conditional statement of two or more banks defaulting given bank i defaulted. With the estimated multivariate GH skewed-t distributions, we can use simulations to compute the risk indicators. We use 10,000,000 simulations at each time t and count the number of banks under stress. As we obtain the simulations directly, we can compute the conditional and unconditional default probabilities. Alternatively if we use the GAS-Equicorrelation model, we can analytically calculate these measures under the LLN approximation suggested in Section 4. The analytical calculation is fast and less cumbersome than the simulation method.

From Figure 3, we see that the dynamic patterns of the risk indicators are very similar irrespective of the computation method used. The Banking Stability measures simulated/calculated from different correlation models are close to each other. The LLN approximated risk measure somewhat understates the risk in normal times and overestimates the risk in crisis times after the year 2008. This is because the number of banks is as small as 10 in our current setting, which makes the LLN approximation less accurate. Figure 4 plots the Systemic Risk Measure proposed in Section 4. The simulated (SIM) measure is computed with the straightforward simulation method and the correlation matrix is driven by the estimated GAS model in Result 1. The LLN approximated Systemic Risk Measure is calculated analytically based on the dynamic Equicorrelation estimates. We see the difference in the SRM between these two methods. The approximated SRM with the conditional Law of Large Numbers is always lower than the simulated SRM, but the pattern over time is similar. If we look at the average of the approximated indicator in the last panel, we see a break around the year 2002 in the mean for the analytical SRM. This may be attributed to the introduction of the Euro as a common currency, which tightened the interconnectedness of the European banks.

5.2 European large financial institutions

The task becomes more challenging with a few European large financial institutions. These financial institutions are large and possibly systemically important, as their failure would likely spread and have adverse implications for financial markets or other financial institu-

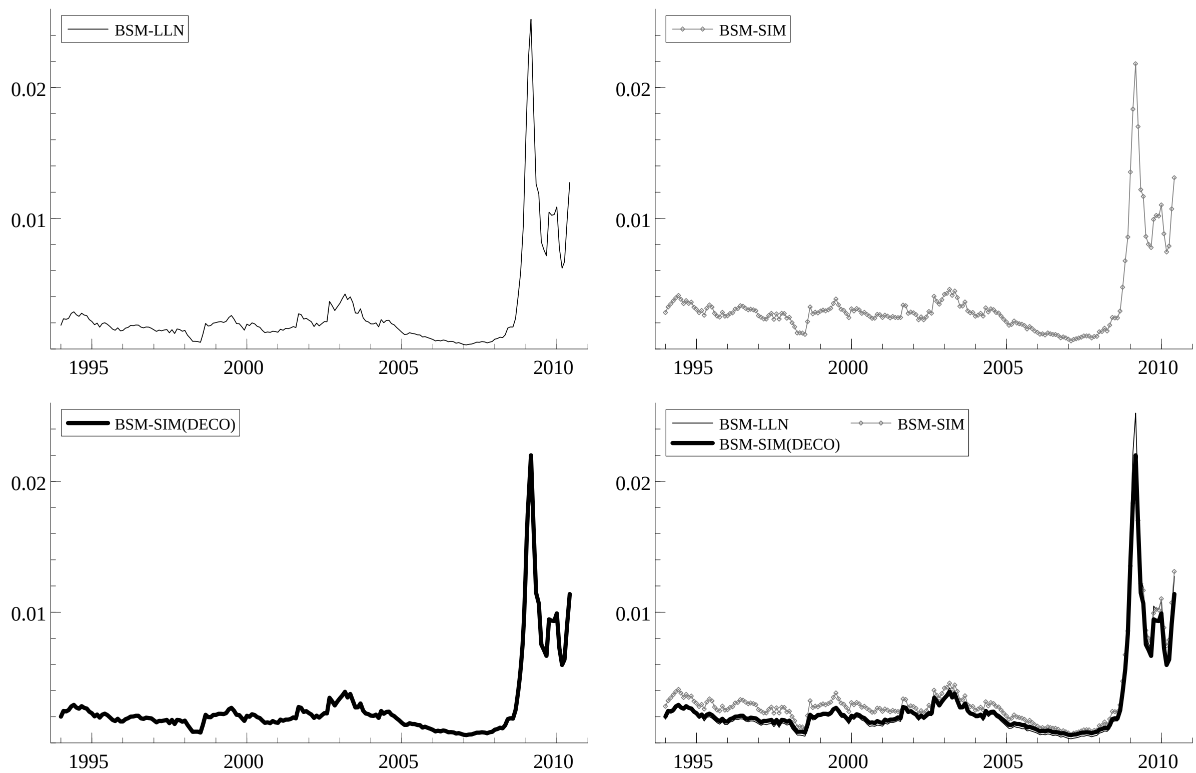

Figure 3: The Banking Stability Measures: a comparison

The Banking Stability Measure constructed from the Dynamic GH skewed-t models. A comparison study is provided here with two different correlation assumptions. The top left and bottom left panel contains the BSM with Dynamic Equicorrelation, but the top one is calculated with the analytical computation and the other one is simulated. The top right plot shows the simulated BSM with the full model correlation result. These measures are defined as the probability of three or more firm defaults.

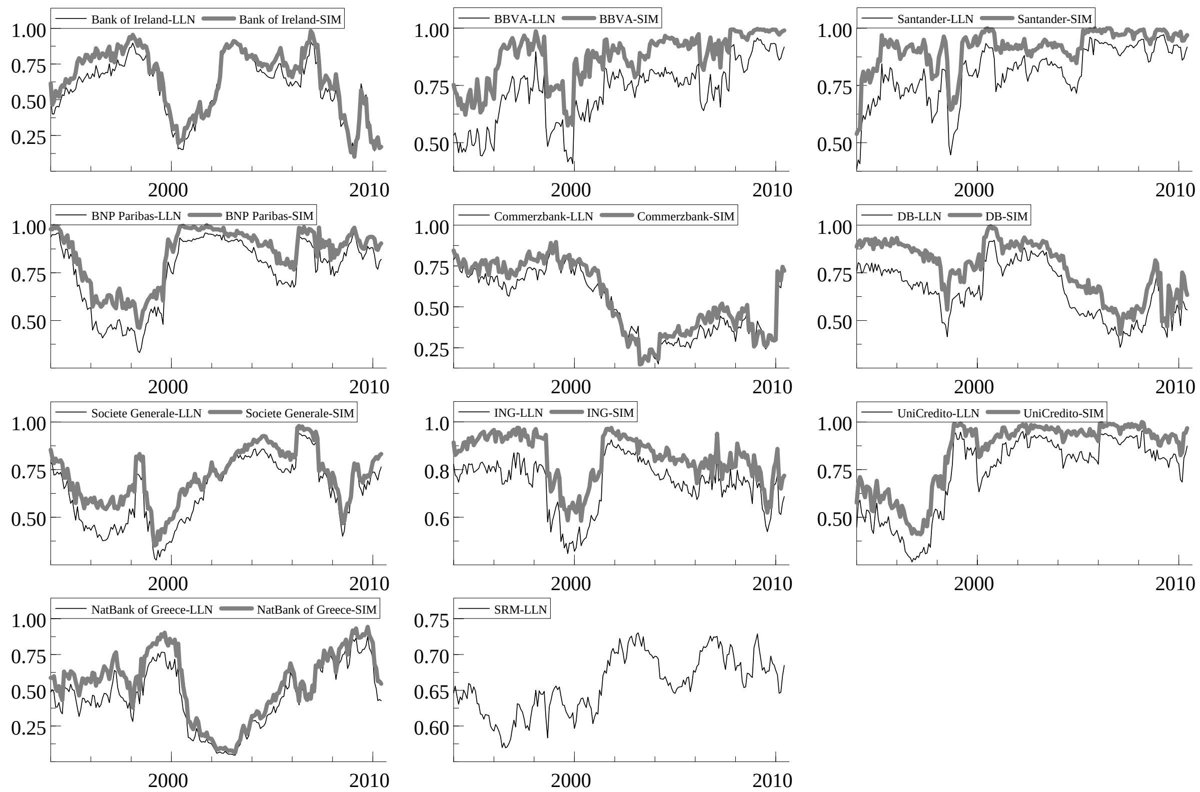

Figure 4: The Systemic Risk Measures: a comparison

The Systemic Risk Measure constructed from the Dynamic GH skewed-t models. We show the result of simulated SRM with correlation estimates from a Full GAS model, as well as the LLN approximated SRM from a Dynamic Equicorrelation model. The last panel contains the average of the SRM measure over all firms. SRM is defined as the probability of two or more firms defaulting given firm i failing.

tions operating within the system.

The datasets we use are monthly equity returns from 73 financial institutions. These institutions are European banks, insurance companies and investment companies. In Table 3, we provide a full list of the names in our sample. The sample skewness and kurtosis for each time series is also included in the table. Most equity return series exhibit negative skewness and fat-tailness.

Table 3: Sample Skewness and Kurtosis Statistics.

Descriptive statistics for the CRSP stock returns between January 1970 and June 2010. All observations are monthly log returns. All names are large European financial firms including banks, insurance companies and investment firms.

Name	Skewness	Kurtosis	Name	Skewness	Kurtosis
ACKERMANS & VAN HAAREN	-0.10	3.92	DEUTSCHE BANK (XET)	-0.36	6.55
AEGON	-1.13	6.75	DEUTSCHE BOERSE (XET)	-0.30	3.98
AGEAS (EX-FORTIS)	-3.78	30.21	DEUTSCHE POSTBANK (XET)	-1.39	8.42
ALLIANZ (XET)	-0.58	5.77	DEXIA	-0.83	7.56
ALLIED IRISH BANKS	-2.16	13.41	EFG EUROBANK ERGASIAS	-0.21	5.19
ALPHA BANK	-0.42	4.36	ERSTE GROUP BANK	-0.61	9.86
GENERALI	-0.83	5.40	EURAZEO	-0.45	5.00
ATRIUM EUROPEAN RLST.	-0.32	10.60	FONCIERE DES REGIONS	-0.85	8.33
AXA	-1.05	6.67	GECINA	-0.33	7.49
AZIMUT HOLDING	-0.26	3.43	GBL NEW	-0.85	5.06
BANK OF IRELAND	-0.32	13.30	SOCIETE GENERALE	-0.72	4.72
BANKINTER 'R'	0.09	4.97	HANNOVER RUCK. (XET)	-0.85	6.65
BANCA CARIGE	-1.36	8.54	ICADE	-0.29	3.76
BANCA MONTE DEI PASCHI	-0.95	5.76	IMMOFINANZ	-2.75	19.24
BANCA POPOLARE DI MILANO	-0.61	4.37	ING GROEP	-1.36	9.58
BANCA PPO.DI SONDRIO	-0.28	3.71	INTESA SANPAOLO	-0.96	5.40
BANCA PPO.EMILIA ROMAGNA	-1.02	7.33	KBC GROUP	-0.99	9.54
BBV.ARGENTARIA	-0.33	4.47	KLEPIERRE	-0.97	6.20
BANCO COMR.PORTUGUES 'R'	-0.50	4.10	MAPFRE	-0.40	4.94
BANCO DE VALENCIA	-0.25	4.27	MARFIN INV.GP.HDG.	0.19	3.78
BANCO ESPIRITO SANTO	-1.03	6.19	MEDIOBANCA	0.11	4.36
BANCO POPOLARE	-0.99	8.06	MUENCHENER RUCK. (XET)	-0.37	10.20
BANCO POPULAR ESPANOL	-0.34	6.42	NATIONAL BK.OF GREECE	-0.37	4.70
BANCO DE SABADELL	-0.24	3.99	NATIXIS	0.26	8.21
BANCO SANTANDER	-0.66	4.73	BANK OF PIRAEUS	-0.45	3.62
BNP PARIBAS	-0.66	6.56	POHJOLA PANKKI A	-1.77	13.97
BOLSAS Y MERCADOS ESPANOLES	-0.07	3.85	RAIFFEISEN INTL.BK.HLDG.	-0.99	5.89
CATTOLICA ASSICURAZIONI	-0.35	5.45	SAMPO 'A'	-0.48	3.63
CNP ASSURANCES	-0.65	4.01	SCOR SE	-2.58	17.83
COFINIMMO	-1.43	7.86	SOFINA	-0.86	5.01
COMMERZBANK (XET)	-0.92	6.31	UBI BANCA	-0.86	7.05
CIE.NALE.A PTF.	-0.35	3.20	UNIBAIL-RODAMCO	-0.78	3.54
CORIO	-0.84	4.47	UNICREDIT	-0.45	7.71
CREDIT AGRICOLE	-0.66	4.17	VIENNA INSURANCE GROUP A	-0.66	13.37
CREDITO VALTELLINES	0.16	5.09	WENDEL	-0.98	4.58
CRITERIA CAIXACORP	-0.73	4.05	WERELDHAVE	-0.19	2.69
DELTA LLOYD GROUP	-0.32	1.70			

The sample covers the period between January 1992 and June 2010. But the length of time series differs for each financial institution. The longest time series contains 488 observations and the shortest one has 10 observations. We modified our model to adapt to this structure. We assume the time-varying equicorrelation matrix is driven by one common factor that follows the GAS process. The correlation between two institutions starts to load this dynamic factor once the equity returns become available for both names. So the size of the correlation matrix is also changing over time and reaches 73 at the maximum. There are two approaches to compute the stability measure for this large dimensional dataset. One is the simulation method proposed in Section 5.1. The drawback is that it takes a long time to generate enough simulations for all possible stressed scenarios. The alternative way is to use the law of large numbers (LLN) rule to approximate the probability, as in Section 4. This approach is numerically easier and still sensible if the main purpose of the study is a joint risk analysis as demonstrated in the previous subsection.

We assume our 73 institutions form a homogenous portfolio. That means all the institutions have the same skewness and kurtosis coefficients γ and ν in the multivariate dynamic distribution for their equity returns. With the volatilities estimated from marginal GAS-GH skewed-t model, we standardize the equity returns and focus on the modeling of dynamic correlations. A multivariate GHST distribution is estimated with the equicorrelation restriction. The parameter estimation results are shown in Table 4. The correlation coefficient plotted in Figure 5 hovers around 0.3 over time. Compared with a rolling window correlation series (the window size is 12), the GAS equicorrelation is more persistent over time. But the means of these two correlation series are similar.

Table 4: Estimation Results Part II

The parameter estimates in the GAS-GHST Equicorrelation model. These models are estimated with the filtered returns data. The sample covers the period between January 1970 and June 2010.

	GAS Equicorrelation Model				
	A	B	ω	γ	Log-lik
Parameter	0.239	0.897	-1.214	-0.034	-9502.36
Standard Error	(0.181)	(0.124)	(0.040)	(0.010)	

We compute the financial risk measures analytically given the multivariate GHST model and the probability of default from the expected default frequency (EDF) of Moody's KMV. We numerically evaluate the integral (44) to compute the Banking Stability Measure, defined as the probability of more than 10% financial institutions defaulting. The risk measure is plotted in Figure 5. From the figure, the LLN result for the default probability does not

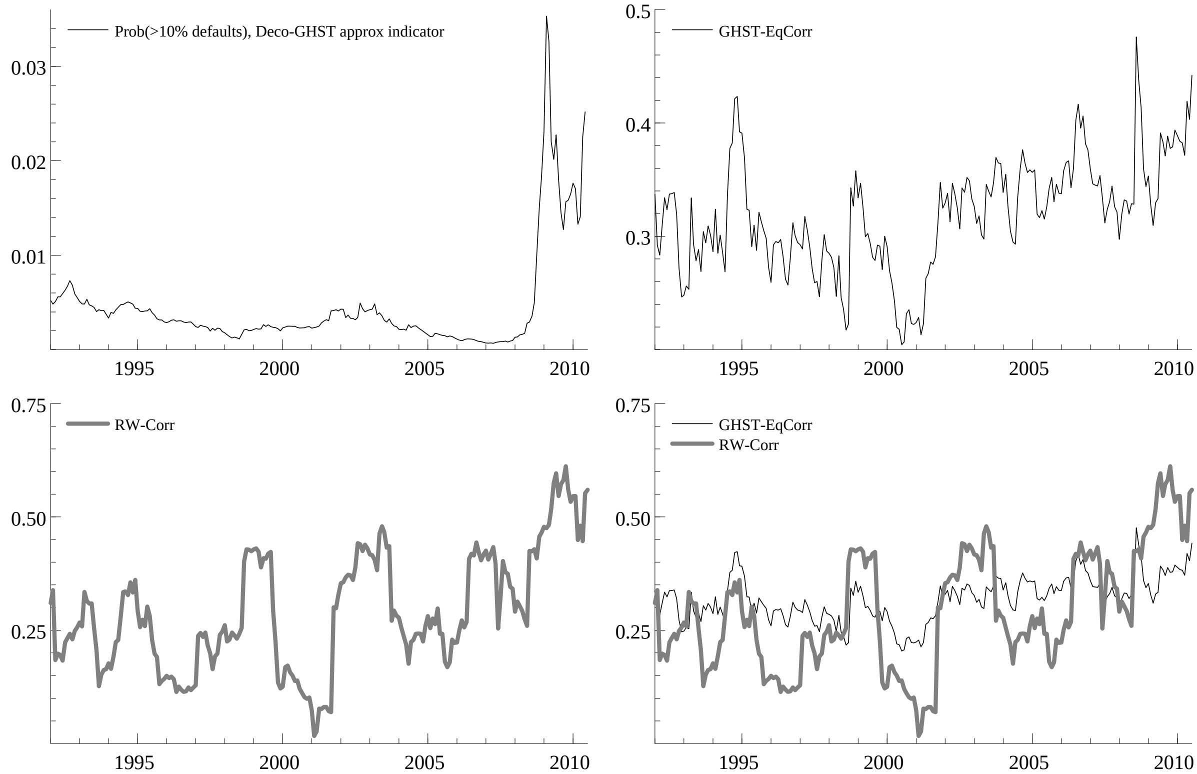

Figure 5: The Banking Stability Measures in whole sample

The Banking Stability Measure defined as the probability of more than 10% firms defaulting under the Law of Large Numbers approximation result. The upper-right panel show the dynamic correlation estimated in the GAS-Equicorrelation model. And the bottom-left panel plots the average of pairwise rolling window correlation coefficients. As a comparison, the bottom-right panel shows the two correlation estimates jointly.

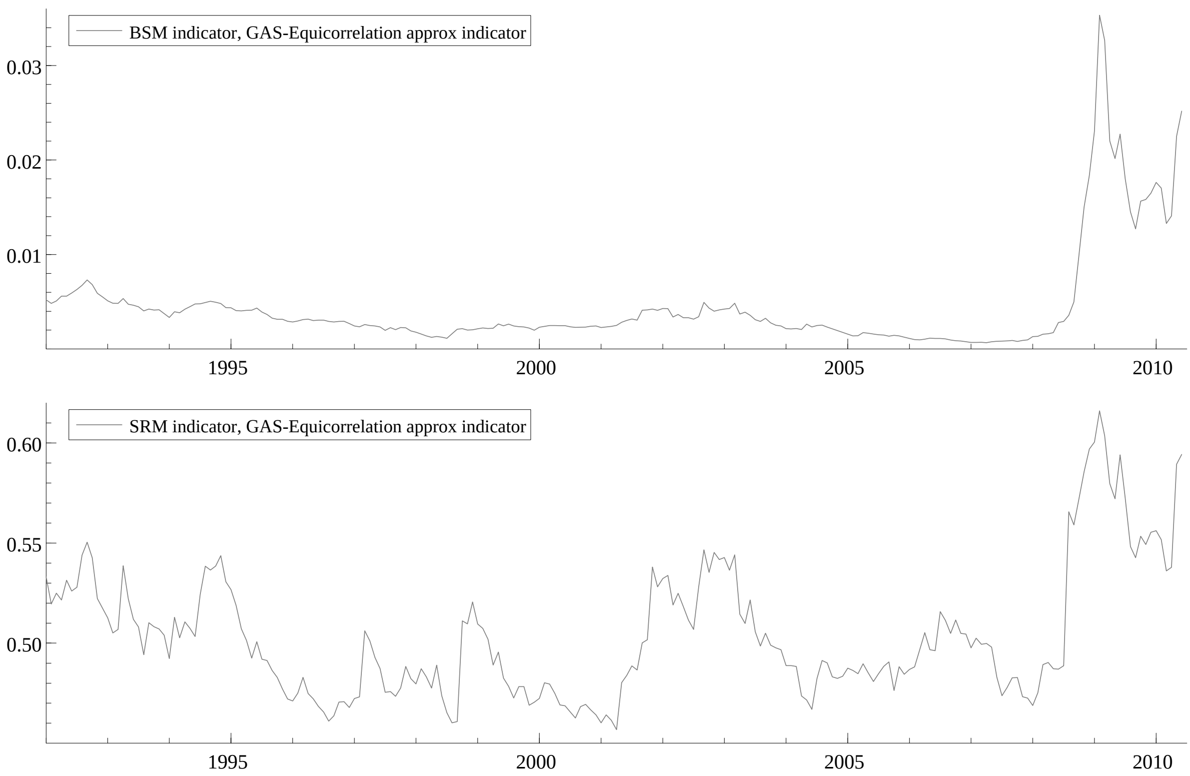

Figure 6: The Banking Stability Measure and Systemic Risk Measure

The Banking Stability Measure (BSM) and Systemic Risk Measure (SRM) under the LLN approximation from the GAS-GHST Equicorrelation model. The BSM indicator is defined as the probability of more than 10% firms defaulting at time t. The SRM indicator is the average of the default probability of more than seven other firms default conditional on firm i defaulting.

move too much before 2008. But it appears that the period of 2008-2010 is quite special: the failure probability increases to more than five times the historical mean. We also compute the same measure with the simulation method. The approximated risk indicator is the same as the simulated one. So we did not include that in the graph. We plot the LLN approximated risk indicators, the Banking Stability Measure and the Systemic Risk Measure in Figure 6. From the graphs, we see the large influence of the recent financial crisis, which drives up the two risk measures in that period. Note that the systemic risk indicator shoots up to 0.60 around the failure of Lehman Brothers.

6 Conclusion

In this paper, we develop the dynamic GHST model with GAS-Equicorrelation or block GAS-Equicorrelation structure. These models are applicable to large dimensional problems. We also propose two risk measures with a large panel of multiple European financial institutions. The Banking Stability measure we developed indicates the joint default risk in the system. The Systemic Risk Measure takes the average of conditional default probabilities to test the interconnectedness of the financial system. The full dynamic multivariate model with the GH skew-t distribution is used to simulate the possible distress scenarios for the banks. Based on the Monte Carlo simulation, we can analyze the joint and conditional credit risk in individual financial institutions. Another risk measuring model originates from the conditional Law of Large Numbers approximation method. With the application of a Dynamic Equicorrelation model in a large system of financial firms, the approximated risk indicator provides a good measure of credit risks for an unbalanced large panel.

We are currently studying the explanatory power of some commonly used economic variables (VSTOXX index, Euribor-EONIA spread and European stock market index) to explain systemic correlation dynamics. By introducing these new variables in our dynamic system, the correlation becomes less persistent compared to the pure GAS dynamic model. The residual GAS factor decreases due to the explanatory power of the extra economic variables. It appears that we still miss one or a few more factors to explain the variation in correlation dynamics. Moreover, we might miss a few important firm specific variables, such as the leverage ratio. The current model also enable us to measure the systemic risk contribution of each bank by looking at the conditional probability in the multivariate GH skewed-t distribution. We leave this for future research.

Appendix: the dynamic GAS-Equicorrelation model

The GH skewed-t distribution is a subclass of the GH distribution family which preserves much of the flexibility of GH distribution, but with less parameters. With the observed stock return for bank i defined as y_{it} following the GH skewed-t distribution, the model is

$$\begin{aligned} y_t &\sim p(\tilde{\Sigma}_t, \nu, \gamma), && \text{(A1)} \\ \tilde{\Sigma}_t &= L_t(TT')L_t', && \text{(A2)} \\ \Sigma_t &= L_tL_t' = R_t, && \text{(A3)} \end{aligned}$$

where γ collects the skewness parameters and the matrix T satisfies the condition

$$(T'T)^{-1} = \frac{\nu}{\nu-2}\mathrm{I}_N + \frac{2\nu^2}{(\nu-2)^2(\nu-4)}\gamma\gamma'. \tag{A4}$$

The deco-Dynamic-GH model defines the correlation matrix as

$$R_t = (1-\rho_t)I_N\rho_t\ell_N\ell_N'N,\ \rho_t \in (\frac{-1}{N-1}, 1), \tag{A5}$$

where $\Sigma_t = R_t$ is the dynamic conditional correlation matrix we are interested in and $\ell_N \in \mathbb{R}^N$ is a vector of ones. In this model, we define L_t as a symmetric matrix. Further, we parameterize ρ_t as a GAS model

$$\begin{aligned} \rho_t &= \vartheta(f_t), && \text{(A6)} \\ f_{t+1} &= \omega + As_t + Bf_t. && \text{(A7)} \end{aligned}$$

We also define the scale matrix as $\tilde{\Sigma}_t = \tilde{L}_t\tilde{L}_t{}'$. The variable T links these two matrices such that $\tilde{L}_t = L_tT$.

The innovation term s_t is the scaled observation density score as in Zhang et al. (2011).

Note that the matrix L_t is symmetric.

$$
\begin{aligned}
s_t &= \mathcal{S}_t \nabla_t, && \text{(A8)}\\
\nabla_t &= \Psi_t' H_t' \mathrm{vec}\left(w_t \cdot y_t y_t' - \tilde{\Sigma}_t - \left(1 - \frac{\nu}{\nu-2} w_t\right) \tilde{L}_t \gamma y_t'\right), && \text{(A9)}\\
\mathcal{S}_t &= \left\{\Psi_t'(\mathrm{I} \otimes \tilde{L}_t^{-1})'[gG - \mathrm{vec}(\mathrm{I})\mathrm{vec}(\mathrm{I})'](\mathrm{I} \otimes \tilde{L}_t^{-1})\Psi_t\right\}^{-1}, && \text{(A10)}\\
H_t &= ((\mathrm{I}_{N^2} + \mathcal{C}_N)(\Sigma_t \otimes \tilde{\Sigma}_t))^{-1}, && \text{(A11)}\\
\Psi_t &= \frac{\partial \mathrm{vec}(\Sigma_t)'}{\partial f_t}, && \text{(A12)}
\end{aligned}
$$

where $g = \frac{\nu+d}{\nu+2+d}$ and G is defined as in Creal et al. (2011).

From the derivation, it is clear that we have to take inverses and compute the determinants of matrices in a large dimension. If we have the matrices in blocks or in the form of the equicorrelation model, we can obtain the inverse and determinant in analytical form which will help to speed up the computations. To get the matrix L_t in an easy-to-operate form, we have

$$\tilde{L}_t = a_t \mathrm{I}_N + b_t \ell_N \ell_N', \tag{A13}$$

where $a_t = \sqrt{1-\rho_t}$ and $b_t = (\sqrt{1-\rho_t+N\rho_t} - \sqrt{1-\rho_t})/N$. The condition for the correlation matrix to be positive definite does not change.

$$T^{-1} = c\mathrm{I}_N + d\ell_N \ell_N', \tag{A14}$$

where

$$
\begin{aligned}
c &= \frac{1}{\sqrt{\mu_\varsigma}},\\
d &= \frac{\sqrt{\mu_\varsigma} - \sqrt{\sigma_\varsigma^2 \gamma'\gamma + \mu_\varsigma}}{\sqrt{\mu_\varsigma}},\\
\mu_\varsigma &= \frac{\nu}{\nu-2},\\
\sigma_\varsigma^2 &= \frac{2\nu^2}{(\nu-2)^2(\nu-4)}.
\end{aligned}
$$

So we have

$$L_t = \tilde{L}_t T^{-1} = a_t c \mathrm{I}_N + (a_t d + b_t c + N b_t d)\ell_N \ell_N'. \tag{A15}$$

It is straightforward to derive the inverse and determinant as

$$\begin{aligned} L_t^{-1} &= \frac{1}{a_t c}\mathrm{I}_N - \frac{a_t d + b_t c + N b_t d}{a_t c(a_t c + N(a_t d + b_t c + N b_t d))}\ell_N \ell_N', && \text{(A16)} \\ \det(\Sigma_t) &= \det(L_t)^2 = (a_t c)^{2(N-1)}(a_t c + N(a_t d + b_t c + N b_t d))^2. && \text{(A17)} \end{aligned}$$

In the application with the whole sample, it appears that the computation of the scale matrix and score matrix takes too much time. One reason is the inversion of a large $N \times N$ matrix in equation (A11). In order to reduce the burden for calculation, we manage to derive Ψ_t analytically, which would speed up the computational speed.

$$\begin{aligned} \Psi &= \frac{\partial \mathrm{vec}(L_t)}{\partial f_t} = \left(\frac{\mathrm{d}a_t}{\mathrm{d}\rho_t} c\mathrm{vec}(\mathrm{I}_N) + (\frac{\mathrm{d}a_t}{\mathrm{d}\rho_t} d + \frac{\mathrm{d}b_t}{\mathrm{d}\rho_t} c + \frac{\mathrm{d}b_t}{\mathrm{d}\rho_t} N d)\ell_{N^2} \right) \vartheta'(f_t), && \text{(A18)} \\ \frac{\mathrm{d}a_t}{\mathrm{d}\rho_t} &= -\frac{1}{2\sqrt{1-\rho_t}}; && \text{(A19)} \\ \frac{\mathrm{d}b_t}{\mathrm{d}\rho_t} &= \frac{1}{2N}\left(\frac{N-1}{\sqrt{1-\rho_t+N\rho_t}} + \frac{1}{\sqrt{1-\rho_t}} \right). && \text{(A20)} \end{aligned}$$

This does help in getting out the correlation simply. The scale matrix $\mathcal{S}_t$ is the inverse Fisher information matrix from the symmetric t distribution, as explained in Zhang et al. (2011).

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