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Hidden Illiquidity with Multiple Central Counterparties

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Abstract

Regulatory changes are transforming the multi-trillion dollar swaps market from a network of bilateral contracts to one in which swaps are cleared through central counterparties (CCPs). The stability of the new framework depends on the resilience of CCPs. Margin requirements are a CCP's first line of defense against the default of a counterparty. To capture liquidity costs at default, margin requirements need to increase superlinearly in position size. However, convex margin requirements create an incentive for a swaps dealer to split its positions across multiple CCPs, effectively "hiding" potential liquidation costs from each CCP. To compensate, each CCP needs to set higher margin requirements than it would in isolation. In a model with two CCPs, we define an equilibrium as a pair of margin schedules through which both CCPs collect sufficient margin under a dealer's optimal allocation of trades. In the case of linear price impact, we show that a necessary and sufficient condition for the existence of an equilibrium is that the two CCPs agree on liquidity costs, and we characterize all equilibria when this holds. A difference in views can lead to a race to the bottom. We provide extensions of this result and discuss its implications for CCP oversight and risk management.

1. Introduction

Swap contracts enable market participants to transfer a wide range of financial risks, including exposure to interest rates, credit, and exchange rates. But swaps themselves can be risky. They create payment obligations that often extend for five to ten years, and they allow participants to take on highly leveraged positions. Indeed, while its proponents see the multi-trillion dollar swap market as an efficient mechanism for risk management and transfer, critics have long seen it as an opaque threat to financial stability.

Regulatory changes are transforming the swap market. Prior to the financial crisis of 2007–08, nearly all swaps traded over-the-counter (OTC) as unregulated bilateral contracts between swap dealers or between dealers and their clients. In contrast, the 2010 Dodd-Frank Wall Street Reform

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and Consumer Protection Act requires *central clearing* of all standard swap contracts in the United States, and the European Market Infrastructure Regulation (EMIR) imposes the same requirement in the European Union. The new rules also bring greater price transparency to swaps trading.

In an OTC market, when two dealers enter into a swap contract, they commit to make a series of payments to each other over the life of the swap. Each dealer is exposed to the risk that the other party may default and fail to make promised payments. In a centrally cleared market, the contract between the two dealers is replaced by two back-to-back contracts with a central counterparty (CCP). The dealers are no longer exposed to the risk of the other's failure because each now transacts with the CCP.

However, this arrangement takes the diffuse risk of an OTC market and concentrates it in CCPs, potentially creating a new source of systemic risk. So long as all its counterparties survive, the CCP faces no risk from its swaps — its payment obligations to one party are exactly offset by its receipts from another party. But for central clearing to be effective, the CCP needs to have adequate resources to continue to meet its obligations even if one of its counterparties defaults. The disorderly failure of a swap CCP would be a major disruption to the financial system with potentially severe consequences for the broader economy.

As its first line of defense, a CCP collects margin from its swap counterparties in the form of cash or other high-quality collateral. Margin — more precisely, *initial* margin — provides a buffer to absorb losses the CCP might incur at the default of a counterparty. If a dealer defaults, the CCP needs to replace its swaps with that dealer, and it may incur a cost in doing so. The initial margin posted by each counterparty is intended to cover this cost in the event of that counterparty's default.

Because of limited liquidity in the market, the replacement cost is likely to be larger for a large position by more than a proportional amount. If the CCP needs to replace a $1 billion swap, it may find several dealers willing to trade. But if it needs to replace a $10 billion swap, it may find few willing dealers, and those that will quote a price may command a premium to take on the added risk of the position. The consequences of this liquidity effect on margin are the focus of this paper.

An immediate implication of limited liquidity is that a CCP's margin requirements should be convex and, in particular, superlinear in the size of a dealer's position. A seemingly obvious but apparently overlooked point is that this is insufficient. The same dealer may have similar positions at other CCPs. If the dealer goes bankrupt, all CCPs at which the dealer participates need to close out their contracts with the dealer at the same time. The impact on market prices is driven by the combined effect from all CCPs. If each CCP sets its margin requirements based only on the positions it sees (as appears to be the case in practice), it underestimates the margin it needs. This is what we call hidden illiquidity. In fact, we show that the very convexity required to capture illiquidity creates an incentive for dealers to split their trades across multiple CCPs, amplifying the effect.

We next examine the possibility that a CCP can compensate for the impact of positions it does not see by charging higher margin on the positions it does see. We analyze this problem through a model with one dealer, two CCPs, and multiple types of swaps. Given margin schedules from the CCPs, the dealer optimizes its allocation of trades to minimize the total margin it needs to post; given the dealer's objective, the CCPs set their margin schedules to have enough margin to cover the systemwide price impact should the dealer default. An equilibrium is defined by margin schedules that meet this objective.

We derive our most explicit results when price impact is linear (so that margin requirements are quadratic). We characterize all equilibria and show, in particular, that margin requirements at the two CCPs need not coincide. A CCP with a steeper margin schedule gets less volume and therefore needs to compensate more for the volume it does not see, which it does with its steeper margin. However, we also show that a necessary condition for an equilibrium is that the two CCPs agree on the true price impact. Without this condition, we get "a race to the bottom" in which a CCP that views the true price impact as smaller drives out the other CCP.

We extend this result to allow CCPs to select a subset of swaps to clear. On the subset of swaps cleared by both CCPs, the previous result applies. Equilibrium now imposes a further necessary and sufficient condition precluding cross-swap price impacts between swaps cleared by just one CCP and swaps cleared by the other CCP. We also consider extensions that introduce uncertainty to the model.

We obtain partial results in the case of nonlinear price impact with a single type of swap. We observe that the dealer's optimization problem combines the convex marginal schedules of the two CCPs into a single *effective margin* which is the inf-convolution of the individual schedules. A result in convex analysis states that the convex conjugate of an inf-convolution of two convex functions is the sum of the conjugates of these functions. We relate this result to conditions for equilibrium.

The rest of this paper is organized as follows: Section 2 provides some background on central clearing. Section 3 introduces the notion of hidden illiquidity. Section 4 introduces our model and our definition of equilibrium. Section 5 considers the case of linear price impact, including a necessary and sufficient condition for equilibrium and an analysis of what happens when the condition fails to hold. In Section 6, we extend the model to include uncertainty. In Section 7, we analyze nonlinear price impact in the case of a single type of instrument. Section 8 concludes the paper and provides practical implications of our analysis. Most proofs appear in an appendix.

2. Background on Central Clearing

Figure 1 illustrates the difference between an over-the-counter market and a centrally cleared market. In part (i) of the figure, dealers A, B, and C trade bilaterally. They initiate trades directly with each other, and each pair of dealers manages payments on its swaps.

The numbers in part (i) of the figure show hypothetical payments due between dealers. Dealers may have multiple swaps with each other — indeed, the number of contracts would typically be very large — leading to payment obligations in both directions. The total payments due at any point in time may be viewed as a measure of the total counterparty risk in the system. In the

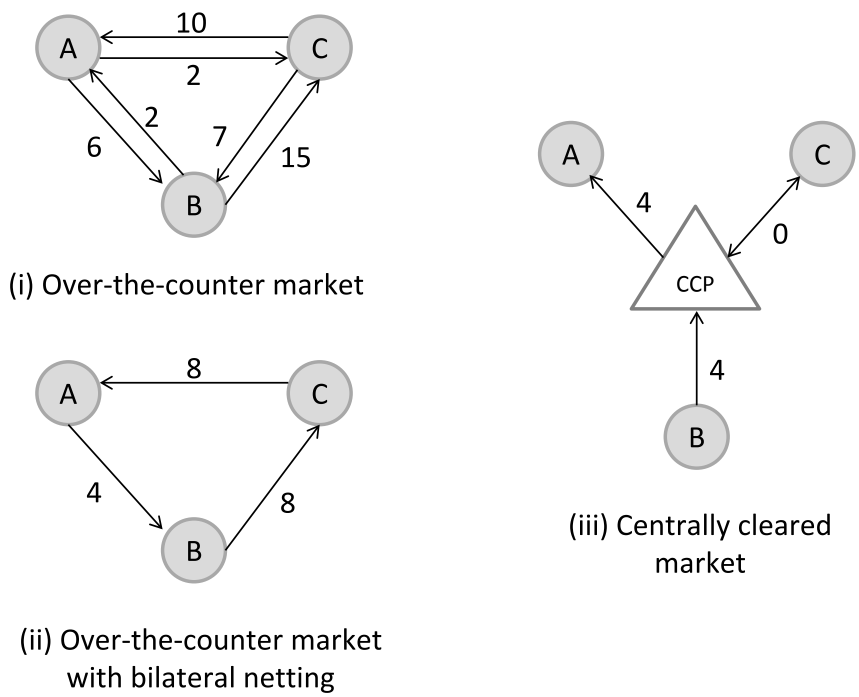


Figure 1: (i) Payment obligations in an OTC market. (ii) Payment obligations after bilateral netting. (iii) Payment obligations in a centrally cleared market.

figure, the total comes to 42.

Bilateral netting between pairs of dealers can greatly reduce total counterparty risk. Part (ii) of Figure 1 shows the result of bilateral netting of payment obligations. Total payments have been reduced to 20. In fact, further netting is still possible — in particular, dealer C makes a net payment of zero. However, further netting would require coordination among all three dealers and cannot be achieved bilaterally.

Part (iii) of the figure illustrates a market with a central counterparty (CCP). After two dealers agree to enter into a swap, their bilateral contract is replaced by two mirror-image contracts running through the CCP.[1] Whatever payments dealer B would have made to dealer A it makes instead to the CCP. The CCP in turn assumes responsibility for making the payments that A would have received from B. With all the contracts from part (i) of the figure running through a single CCP, central clearing achieves maximal netting in part (iii) of the figure, reducing the total payments due to eight. This reduction in systemwide counterparty risk is one of the main arguments for central clearing. Moreover, the CCP theoretically always has a net risk of zero in the sense that the total payments it needs to make on swaps equal the total payments it is owed.

This simple example overstates the benefits of central clearing in several respects. Dealers engaged in different types of OTC swaps — interest rate swaps and credit default swaps, for

[1]Only clearing members of a CCP can trade through the CCP. We will informally refer to the parties to swaps as dealers or clearing members, but strictly speaking a dealer need not be a clearing member and a clearing member need not be a dealer.

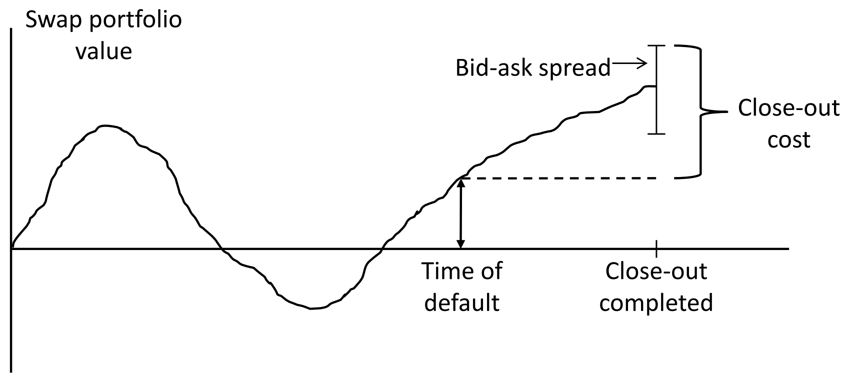


Figure 2: Variation margin covers the value of a clearing member's swap portfolio at the time of default. Initial margin should cover costs the CCP may incur from the time of default to the completion of the close-out of defaulting member's portfolio.

example — can net bilateral payments across all swaps; so, if different types of swaps are cleared through different CCPs, central clearing can actually reduce the total amount of netting. (See Duffie and Zhu (2011) and Cont and Kokholm (2014) for more on this comparison.) Some of the multilateral netting benefit provided by a CCP can be achieved in an OTC market through third-party trade compression services. In both OTC and centrally cleared markets, dealers provide collateral for their payment obligations, which reduces the counterparty risk that remains from any unnetted exposures. With central clearing, the CCP faces risk from the default of a dealer because of the costs it may incur in replacing or unwinding positions after the dealer fails.

This last point motivates our analysis so we discuss it in further detail. To protect itself from the failure of a clearing member, the CCP collects two types of margin payments from each member on at least a daily basis, *variation margin* and *initial margin*. Variation margin reflects daily price changes in a clearing member's swaps. If the market value of the member's swaps decreases, the member makes a variation margin payment to the CCP; if the market value increases, the CCP credits the member's variation margin account. At the time of a clearing member's default, the variation margin collected by the CCP should offset the value of the clearing member's position.

Figure 2, based on a similar figure in Murphy (2012), illustrates the two types of margin. The figure shows the hypothetical evolution of the value of a clearing member's swap portfolio over time, from the perspective of the CCP. The value may be positive or negative. In the figure, the clearing member fails at a time when its swaps have positive value to the CCP. The variation margin held by the CCP allows the CCP to recover this value upon the clearing member's failure.

However, the CCP cannot instantly replace or liquidate the failed member's positions. Suppose, for example, that dealer B in Figure 1 had a single swap, originally entered into with dealer A and subsequently cleared through the CCP. If dealer B fails, the CCP has to continue to meet its payment obligations to dealer A. In order to do so, it needs to replace the position held by B.

Replacing dealer B's position may take several days. During this time, the market value of the position will continue to move, as illustrated in Figure 2. The value of the CCP's claim on dealer B is also the value of dealer A's claim on the CCP. An increase in the market value after B's failure, as illustrated in the figure, represents a loss to the CCP. The initial margin collected by the CCP is intended to protect the CCP from such losses. Moreover, when the CCP transacts it incurs the cost of the bid-ask spread. This cost should also be covered by the initial margin.

For purposes of illustration, Figure 2 shows the change in market value and the bid-ask spread as two separate contributions to the total cost incurred by the CCP. In fact, the two sources of loss are entangled. If the CCP transacts more quickly, buying and selling large positions, it will face lower market risk but incur higher liquidity costs through wider bid-ask spreads. It can try to reduce liquidity costs by breaking the failed member's positions into smaller pieces and replacing them more slowly. In doing so, it faces greater market risk. See Avellaneda and Cont (2013) for an analysis of a CCP's optimal liquidation problem.

Larger transactions face wider bid-ask spreads per dollar traded. As a consequence, liquidity costs increase superlinearly in the size of a position. Initial margin must then also grow superlinearly to cover liquidity costs with high probability. Hull (2012) calls this the size effect.

We will argue, however, that superlinear margin requirements create an incentive for a dealer to split trades across multiple CCPs. If the dealer fails, all CCPs through which it trades will need to replace the dealer's positions at the same time. Their liquidation costs will be driven by the total size of the dealer's positions across all CCPs. If each CCP bases its margin requirements solely on the trades it clears, without considering trades by the same dealer at other CCPs, it will underestimate the margin it needs to cover liquidation costs.

In addition to variation margin and initial margin, clearing members make contributions to a CCP's guarantee fund. If a clearing member defaults, any losses exceeding that member's margin are first absorbed by the member's guarantee fund contribution, then by CCP capital, and then by the guarantee fund contributions of surviving members. However, initial margin is required to cover liquidation costs with 99 percent confidence under U.S. regulations (Commodity Futures Trading Commission, 2011, p. 69368–69370), or 99.5 percent under EMIR (European Commission, 2013, p. 56), so our analysis will focus on the adequacy of the margin collected.

Other work on CCP margins includes Cruz Lopez et al. (2013) and Menkveld (2014), both of whom focus on dependence between the trades of members of a single CCP. Amini et al. (2013) consider the impact of central clearing on overall systemic risk. Capponi et al. (2014) examine concentration in CCP membership. Biais et al. (2012) study the incentives created by loss mutualization in a CCP. Pirrong (2009) provides a detailed critique of central clearing.

3. Hidden Illiquidity

We contrast margin requirements based solely on market risk with requirements that reflect liquidity costs. We assume that the CCP is able to collect variation margin to cover routine daily price changes, so by "margin" we mean initial margin.

We consider a dealer that is a clearing member of K identical CCPs. Each CCP clears m types of swaps. These could be credit default swaps (CDS) on different reference entities or with different terms, or they could be different types of interest rate swaps. A vector $x \in \mathbb{R}^m$ records the dealer's swap portfolio, with the ℓth component of x measuring the size of a dealer's position in swaps of type ℓ, $\ell = 1, \ldots, m$.

To clear a vector of swaps x, each CCP collects margin $f(x)$, for some margin function $f\colon \mathbb{R}^m \to \mathbb{R}_+$ that is common to all CCPs. We allow the dealer to divide the position vector x arbitrarily among the K CCPs, clearing the vector x_i through the ith CCP, with $x_1 + \cdots + x_K = x$. To minimize the total margin it needs to post, the dealer solves

$$\underset{x_1,\ldots,x_K \in \mathbb{R}^m}{\text{minimize}} \left\{ \sum_{i=1}^{K} f(x_i) \;\middle|\; \text{subject to } x_1 + \cdots + x_K = x \right\}. \tag{1}$$

A margin requirement for market risk alone seeks to cover the 99th or 99.5th percentile of a portfolio's change in market value between the time of default and the end of the closeout period indicated in Figure 2, ignoring liquidity costs. The closeout period is typically assumed to be five to ten days. The percentile can be approximated as a multiple of the standard deviation of the change in value over this period. If we let Σ denote the $m \times m$ covariance matrix of price changes for the m types of swaps over the close-out period, then we can define a margin requirement to cover market risk by setting

$$f(x) \triangleq a(x^\top \Sigma x)^{1/2}, \tag{2}$$

for some multiplier a.

With this choice of f, the dealer could optimally clear the entire portfolio x through a single CCP. Sending x/K to each CCP is also optimal, but the dealer receives the full benefit of diversification through a single CCP — there is no incentive for the dealer to split the position. Moreover, if the dealer does split the position, each CCP receives the margin it needs to cover the market risk it faces, assuming a and Σ are chosen correctly.

The margin function in (2) is convex but it scales linearly in position size: for any $x \in \mathbb{R}^m$ and any $\lambda \geq 0$, $f(\lambda x) = \lambda f(x)$. In other words, this f is positively homogeneous. As discussed in the previous section, the margin function needs to increase superlinearly in position size to cover liquidity costs. For example, consider

$$f(x) \triangleq a(x^\top \Sigma x)^{\alpha/2}, \quad \alpha > 1. \tag{3}$$

This margin function yields $f(\lambda x) = \lambda^\alpha f(x)$ for any $x \in \mathbb{R}^m$ and $\lambda \geq 0$, so it does indeed grow superlinearly along the direction of any portfolio vector x. In this case, solving (1) requires clearing an equal portion x/K through each CCP. Superlinear margin creates an incentive for the dealer to distribute the position as widely as possible. More generally, we have the following contrast between two types of margin functions.

Proposition 1. *Suppose the function f satisfies $f(0) = 0$. Then:*

(i) If f has the following two properties,

(a) Subadditivity: $f(x+y) \le f(x) + f(y)$, for all $x, y \in \mathbb{R}^m$,

(b) Positive homogeneity: $f(\lambda x) = \lambda f(x)$, for all $x \in \mathbb{R}^m$, $\lambda \ge 0$,

then any allocation of the form $x_i = b_i x$, with $b_1 + \cdots + b_K = 1$ and $b_i \ge 0$, $i = 1, \ldots, K$, solves (1)*. In particular, clearing the full portfolio x through a single CCP is optimal.*

(ii) If f is strictly convex, then an equal split $x_i = x/K$, $i = 1, \ldots, K$, is the only optimal solution to (1)*. Furthermore, the margin requirement is superlinear in the sense that $f(\lambda x) > \lambda f(x)$, for all $x \in \mathbb{R}^m$, $x \ne 0$, and all $\lambda > 0$.*

Proof. For (i), observe that if (a) and (b) hold, then

$$\sum_{i=1}^{K} f(b_i x) = \sum_{i=1}^{K} b_i f(x) = f(x) = f\left(\sum_{i=1}^{K} x_i\right) \le \sum_{i=1}^{K} f(x_i),$$

for any vector $b \ge 0$ satisfying $b_1 + \cdots + b_K = 1$ and any $x_1, \ldots, x_K \in \mathbb{R}^m$ feasible for (1).

For (ii), if f is strictly convex, then for any $x_1 + \cdots + x_K = x$,

$$\sum_{i=1}^{K} f(x_i) = K \sum_{i=1}^{K} f(x_i)/K \ge K f\left(\sum_{i=1}^{K} x_i/K\right) = K f(x/K).$$

The inequality is strict when the vectors $\{x_i\}$ are not identical. ■

We can say more if we specialize to a price impact formulation of liquidity costs. Suppose f takes the form

$$f(x) \triangleq x^\top F(x), \tag{4}$$

where $F\colon \mathbb{R}^m \to \mathbb{R}^m$ satisfies $F(0) = 0$ and is increasing. Interpret $F(x)$ as the impact on the market price of closing out a position x. Then $x^\top F(x)$ is the cost incurred as a result of this price impact on the portfolio x.

Suppose f in (4) is strictly convex, so the dealer optimally splits its position evenly across CCPs. Each CCP collects $x^\top F(x/K)/K$ in margin. If the dealer fails and all CCPs liquidate their identical positions, the total price impact is $F(x)$, so each CCP incurs a cost of $x^\top F(x)/K$, which is larger than the margin it collected. The strict convexity of f motivates the dealer to "hide" part of its position from each CCP and, moreover, leaves each CCP with insufficient margin.

If all CCPs have the same margin function, they can eliminate the problem by charging

$$f(x) \triangleq x^\top F(Kx).$$

In other words, they can precisely compensate for the hidden illiquidity by overstating the cost of liquidating the positions they clear. Clearing regulations[2] require CCPs to back test their margin

[2]See Commodity Futures Trading Commission (2011, p. 69372–69374) or European Commission (2013, p. 65–66).

requirements against historical data. But this simple result implies that a properly calibrated margin model will understate the required margin, unless each CCP considers the simultaneous effects of other CCPs in its analysis. Although they are lengthy and detailed, procedures for swap CCPs adopted by the Commodity Futures Trading Commission (2011) and the European Commission (2013) do not address the need to consider the effect of a member's default at other CCPs, nor is this point noted in the influential principels set forth by the Committee on Payment and Settlement Systems and the International Organization of Securities Commissions (CPSS-IOSCO, 2012). In Section 5.2, we will see that compensating for the effects of other CCPs may be difficult if the CCPs have different margin models and, more importantly, different views on price impact.

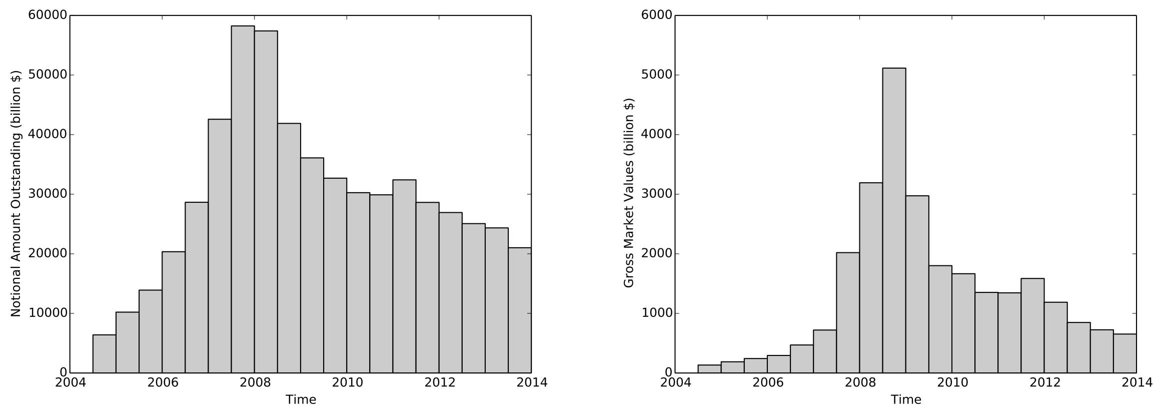
In practice, a dealer may have considerations other than margin minimization in making its clearing decisions. Partly for this reason, we will introduce some uncertainty in the dealer's allocation in Section 6. Two specific constraints deserve comment. First, clearing a swap through a CCP requires that both parties be members of the CCP; in fact, the lists of clearing members for swaps CCPs have substantial overlap, with nearly all the major derivatives dealers members of the major swaps CCPs, particularly for CDS clearing. Second, clearing members clear trades for clients as well as for their own accounts, and this limits their ability to subdivide positions. Such constraints may prevent a dealer from allocating trades uniformly but they do not remove the incentive for the dealer to split positions to the extent possible.

The margin models used by individual CCPs are proprietary. However, the following excerpt from an industry magazine (Ivanov and Underwood, 2011, p. 32) supports our analysis. The article describes the margin methodology at ICE Clear Credit, the largest CCP for credit default swaps:

> *"For portfolio/concentration risks, large position requirements, also known as concentration charges, apply to long and short protection positions that exceed predefined notional threshold levels. The concentration charge threshold reflects market depth and liquidity for the specific index family or reference entity. Positions that exceed selected thresholds are subject to additional, exponentially increasing, initial margin requirements. The accelerated initial margin creates the economic incentive to eliminate large positions."*

Whether the model literally uses an exponential margin function or if this term is used informally to refer to a superlinear increase is unclear.

We should also comment on the degree of liquidity in swaps markets. The most liquid interest rate swaps and index CDS are already centrally cleared. As new types of contracts migrate to CCPs, they are inevitably less liquid, particularly at the outset. Swaptions and inflation swaps have been proposed for central clearing but are far less liquid than standard interest rate swaps. Even among index CDS, off-the-run indexes are significantly less liquid than their on-the-run versions. Each index CDS trades at multiple maturities, and liquidity is much lower at maturities other than five years. Chen et al. (2011) provide a detailed analysis of liquidity in CDS transactions using supervisory data. We make some observations using public data.



(a) Notional amount outstanding

(b) Gross market value

Figure 3: Aggregate CDS market statistics (2005–2013).

Figure 3 shows the notional amount outstanding and gross market value of CDS from 2005 to 2013, as reported by the Bank for International Settlements. Both measures show declining liquidity in the CDS market following the financial crisis. Higher bank capital requirements for derivatives have contributed to this trend.

Figure 4 shows the distribution of the average number of trades per day for all single-name CDS, as publicly reported on the Depository Trust Clearing Corporation's website. The figure shows data for the first quarter of 2013. The vast majority of contracts trade at most a few times per day.

Figure 5 shows the distribution of bid-ask spreads for one-year and five-year CDS, as reported by Markit Group Ltd. The figures show the bid-ask spreads for all single-name contracts for all days in 2013, except that we dropped the top 10 percent (the widest spreads) in both cases. The distribution for five-year contracts shows large spikes near five and ten basis points. For the one-year contracts the spreads are much wider, reflecting the lower liquidity at that maturity.

4. Model

We now turn to a setting with $K = 2$ CCPs. We assume that both CCPs clear a universe of m types of swaps. We consider a dealer that is a clearing member of both CCPs and whose portfolio is described by the vector $x \in \mathbb{R}^m$.

We will measure the liquidation costs associated with a portfolio using price impact functions, defined as follows:

Definition 1 (Price Impact Function). *A price impact function is a function $F\colon \mathbb{R}^m \to \mathbb{R}^m$ satisfying the following conditions:*

(i) $F(0) = 0$,

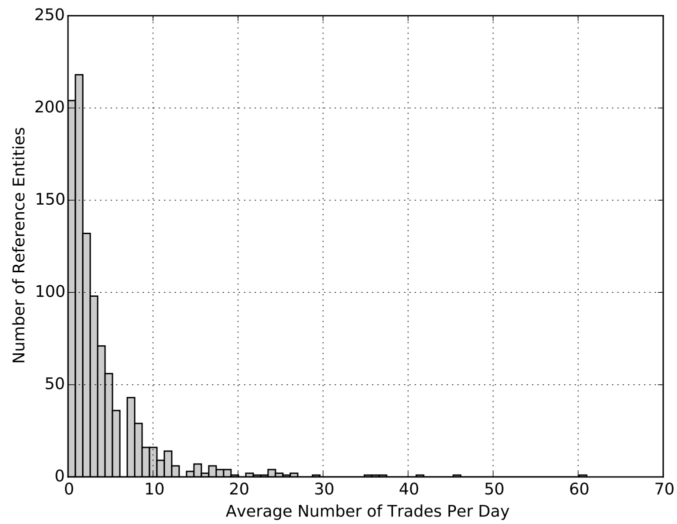
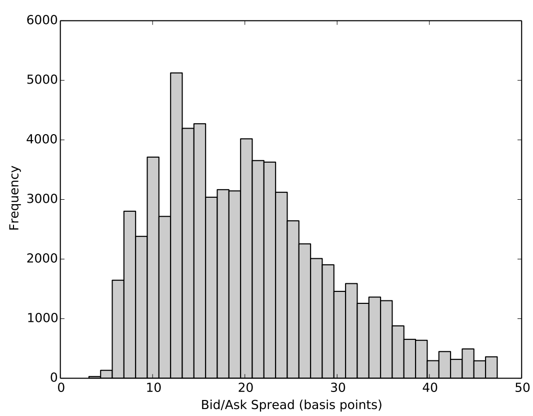
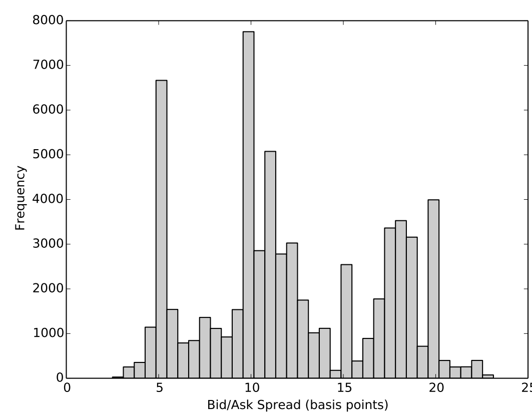


Figure 4: Histogram of average number of daily CDS trades per reference entity (Q1, 2013).



(a) 1 year tenor



(b) 5 year tenor

Figure 5: Histogram of bid-ask spreads for CDS (2013).

(ii) F is differentiable,

(iii) the map $x \mapsto x^\top F(x)$ is strictly convex over $x \in \mathbb{R}^m$.

Here, $F(x)$ captures the vector of price changes that would occur given the liquidation of the portfolio x. Specifically, the ℓth component of the vector $F(x)$ represents the price change to swap ℓ given the liquidation of a portfolio x. Condition (i) requires that if no portfolio is liquidated, then there is no price impact. Condition (ii) will be convenient for technical reasons. Condition (iii) requires that the margin costs associated with the liquidation of a portfolio be increasing with the portfolio size.

We assume that the ith CCP believes that price impact is given by a price impact function $G_i \colon \mathbb{R}^m \to \mathbb{R}^m$. We further assume that the ith CCP charges margin as a function of only the portfolio $x_i \in \mathbb{R}^m$ cleared there by the clearing member. This is done according to an alternative price impact function $F_i \colon \mathbb{R}^m \to \mathbb{R}^m$. In other words, for clearing the portfolio x_i, the ith CCP charges initial margin according to the schedule

$$f_i(x_i) \triangleq x_i^\top F_i(x_i).$$

The clearing member will divide the overall portfolio x in order to minimize the total initial margin outlay. Given margin schedules $\{f_1, f_2\}$, this involves solving the optimization problem

$$f_{\mathsf{eff}}(x) \triangleq \underset{x_1, x_2 \in \mathbb{R}^m}{\text{minimize}} \ \{ f_1(x_1) + f_2(x_2) \mid \text{subject to } x_1 + x_2 = x \}. \tag{5}$$

Here, the optimal value $f_{\mathsf{eff}}(x)$ is the *effective margin* function experienced a clearing member that optimally divides its portfolio across the CCPs.

Given the liquidation of the portfolio x, each CCP should ensure that enough margin is collected to cover liquidation costs. Given that the ith CCP believes that the price movement from the liquidation of the overall portfolio will be given by the vector $G(x)$, CCP i will incur liquidation costs of $x_i^\top G_i(x)$ on the sub-portfolio x_i it clears. Therefore, for CCP i to collect sufficient margin, it is necessary that

$$x_i^\top F_i(x_i) \geq x_i^\top G_i(x). \tag{6}$$

We will assume that the market is competitive, so the CCPs seek to collect no more initial margin than is necessary to cover liquidation costs. In other words, we will replace the inequality in (6) with equality.

Combining the various considerations described above, we define an equilibrium between the clearing member, which seeks to minimize its margin requirements, and the CCPs, which seek to collect sufficient margin to cover liquidation costs, as follows:

Definition 2 (Equilibrium). *Given price impact beliefs G_1, G_2 for the two CCPs, an equilibrium (F_1, F_2, x_1, x_2) is defined by*

1. allocation functions $x_i \colon \mathbb{R}^m \to \mathbb{R}^m$, for $i \in \{1, 2\}$, *2. price impact functions $F_i \colon \mathbb{R}^m \to \mathbb{R}^m$, for $i \in \{1, 2\}$,*

satisfying, for each portfolio $x \in \mathbb{R}^m$,

1. $(x_1(x), x_2(x))$ is an optimal solution to the clearing member's problem (5)*,*

2. each CCP i collects initial margin to meet its true price impact beliefs, i.e.,

$$x_i(x)^\top F_i(x_i(x)) = x_i(x)^\top G_i(x), \quad \textit{for } i \in \{1, 2\}.$$

Definition 2 makes explicit the functional dependence of the allocations x_1 and x_2 on the portfolio x. In what follows, we will sometimes suppress this dependence for notational convenience.

5. Linear Price Impact

We first consider the case of *linear price impact functions*, where we require that the price impact functions associated with each CCP satisfy

$$F_i(x) = F_i x, \quad G_i(x) = G_i x, \tag{7}$$

for some matrices $F_i, G_i \in \mathbb{R}^{m \times m}$. Without loss of generality, we will require that the matrices F_i, G_i be symmetric.[3] Moreover, in order to satisfy Part (iii) of Definition 1, we require that $F_i, G_i \succ 0$, i.e., that the matrices are positive definite.

Given linear price impact (7), the total margin charged by each CCP i takes the form

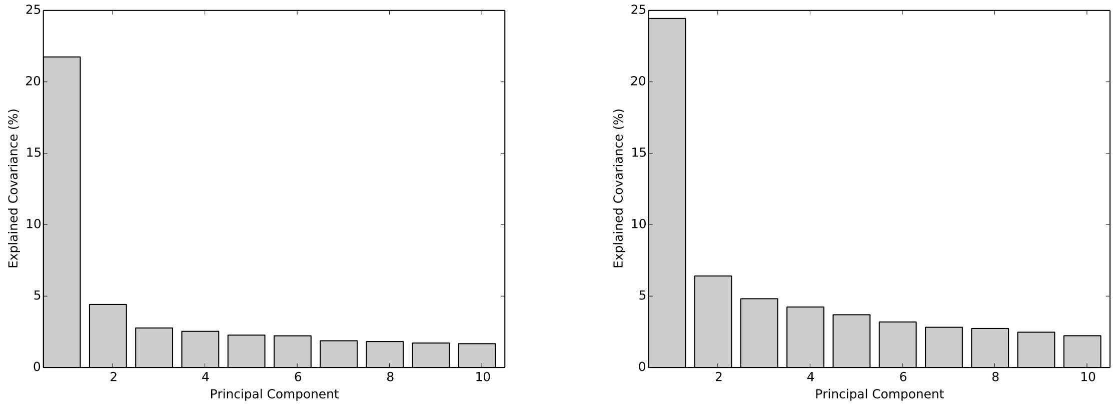
$$f_i(x) = x^\top F_i x,$$

i.e., the CCP margins charged are quadratic in the position cleared. This is a multivariate version of the Kyle (1985) model, in which price impact is linear and the total liquidation costs are quadratic.

A linear price impact model accommodates *cross-price impact*: the (k, ℓ) entry of a linear price impact matrix captures the effect of liquidating the ℓth instrument on the price of the kth instrument. Cross-price impact is important in situations where transactions in one swap propagate to the prices of other swaps. This can occur for supply/demand reasons (e.g., when similar instruments function as partial substitutes) or for informational reasons (e.g., when the underlying fundamental values of related instruments are correlated). For example, CDS for different firms in the same sector can be impacted by common liquidity or price shocks, as are CDS for the same reference entity across various tenors, or CDS for different series of a common index.

Direct estimation of price impact functions requires detailed transaction data and can be quite

[3]For any matrix $F \in \mathbb{R}^{m \times m}$, $x^\top F x = x^\top (F + F^\top) x/2$ for all $x \in \mathbb{R}^m$. Hence, if a price impact matrix F is non-symmetric, we can replace it with its symmetrization $(F + F^\top)/2$ without changing the resulting margin function.



(a) Financial CDS ($n = 326$) (b) Sovereign CDS ($n = 93$)

Figure 6: Variance explained by the first 10 principal components of covariance matrices.

challenging.[4] To get a rough indication of the potential for cross-price impact, we can examine comovements in credit default swaps. Figure 6 shows the variance explained by the first 10 principal components of the covariance matrices of daily CDS returns for financial institutions (left) and sovereigns (right). In both cases, a relatively small number of principal components explains a significant fraction of total variance. This suggests significant cross-price impact within each sector.

5.1. Equilibrium Characterization

In the case of linear price impact functions, the following theorem characterizes possible equilibria:

Theorem 1. *A necessary and sufficient condition for the existence of an equilibrium with linear price impact functions is that the two CCPs have common views on market impact, i.e., that* $G_1 = G_2 \triangleq G$.

In this case, all equilibria are determined by the symmetric, positive definite solutions $F_1, F_2 \in \mathbb{R}^{m \times m}$ *to the equation*

$$G^{-1} = F_1^{-1} + F_2^{-1}. \tag{8}$$

Theorem 1 generates two important insights. First, in order for an equilibrium to exist, the CCPs must agree on the true price impact G. In Section 5.2, we will show that different beliefs about the true price impact can create a "race to the bottom" in which one CCP is driven out of the market.

The second insight of Theorem 1 is that the CCPs need not charge the same margin in equilibrium. There are many possible equilibria, corresponding to solutions of (8). To interpret (8), note

[4]See Fleming and Sarkar (2014) for an analysis of the failure resolution of Lehman Brothers, including its cleared swaps.

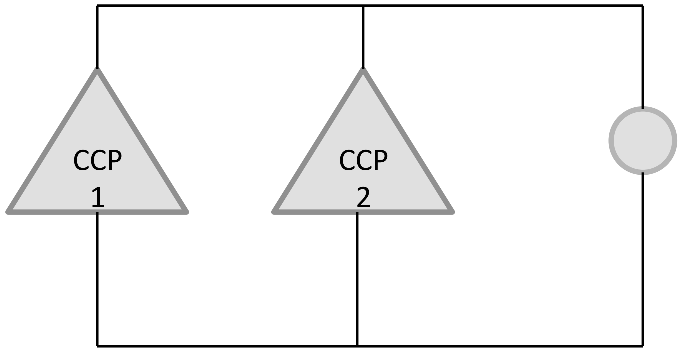


Figure 7: Margin requirements combine like resistors connected in parallel.

that, in the present setting, the clearing member's problem takes the form

$$\begin{aligned} f_{\mathsf{eff}}(x) &\triangleq \underset{x_1,x_2\in\mathbb{R}^m}{\text{minimize}} \; \left\{ x_1^\top F_1 x_1 + x_2^\top F_2 x_2 \;\middle|\; \text{subject to } x_1 + x_2 = x \right\} \\ &= \underset{x_1\in\mathbb{R}^m}{\text{minimize}} \; x_1^\top F_1 x_1 + (x-x_1)^\top F_2 (x - x_1) \\ &= x^\top \left(F_1^{-1} + F_2^{-1} \right)^{-1} x. \end{aligned}$$

Under condition (8), then, we have that $f_{\mathsf{eff}}(x) = x^\top G x$. In other words, the equilibrium condition is equivalent to the requirement that the effective margin experienced by an optimizing clearing member correspond to the margin that would be charged by a single CCP under the common price impact belief G.

A special case of this equilibrium would be

$$F_1 \triangleq \frac{G}{\alpha}, \quad F_2 \triangleq \frac{G}{1-\alpha}, \quad \alpha \in (0,1).$$

When $\alpha = 1/2$, each CCP charges according to twice its true belief, and each clears half of the clearing member's portfolio. This corresponds to the equilibrium discussed in Section 3. If $\alpha < 1/2$, CCP 1 will attract less than half of the portfolio because it has a higher margin charge, so it needs to compensate more for the part of the portfolio it does not see, which it precisely accomplishes through its higher margin charge.

Notice that, in out setting, $G^{-1}\Delta p$ is the size of the portfolio required to achieve a price movement $\Delta p \in \mathbb{R}^m$. In this way, G^{-1} is analogous to the "market depth" of Kyle (1985). Thus Theorem 1 can be interpreted as follows: in an equilibrium we require that the two CCPs agree on the true market depth, and that the total depth provided by the two CCPs match the true depth.

Further, the operation $(F_1, F_2) \mapsto (F_1^{-1} + F_2^{-1})^{-1}$ is called the "parallel sum" of matrices in Anderson and Duffin (1969) and a subsequent literature. The name is based on an analogy with how resistors combine when connected in parallel in a circuit. To make the analogy in our setting (see Figure 7), identify the price impact used by each CCP with resistance, identify the size of the clearing member's trade with current, and identify the total price impact with voltage.

5.2. Race to the Bottom

Theorem 1 establishes that there can be no equilibrium with linear price impact functions if the CCPs have differing beliefs of price impact. In order to provide intuition for why this is the case, it is useful to analyze the best response dynamics between competing CCPs in this setting.

Specifically, consider a discrete time setting indexed by $t = 0, 1, \ldots$, where CCPs sequentially update their margin requirements as follows:

1. At time $t = 0$, each CCP i sets margins according to its initial beliefs by setting $F_i(0) \triangleq G_i$.

2. At each time $t \geq 0$, given margins specified by symmetric, positive definite impact matrices $\big(F_1(t), F_2(t)\big)$:

   (a) The clearing member computes the optimal allocation $\big(x_1(t), x_2(t)\big)$ by solving (5) assuming price impact matrices $\big(F_1(t), F_2(t)\big)$ and gets

   $$x_1(t) = \big(F_1(t) + F_2(t)\big)^{-1} F_2(t) x, \quad x_2(t) = \big(F_1(t) + F_2(t)\big)^{-1} F_1(t) x. \tag{9}$$

   (b) Given the clearing member's allocation $\big(x_1(t), x_2(t)\big)$, CCP 1 sets its price impact matrix $F_1(t+1)$ for the next period to ensure that it would get sufficient margin for the present allocation by solving

   $$x_1(t)^\top G_1 x = x_1(t)^\top F_1(t+1) x_1(t).$$

   Using (9), we have that

   $$x^\top F_2(t)\big(F_1(t)+F_2(t)\big)^{-1} G_1 x = x^\top F_2(t)\big(F_1(t)+F_2(t)\big)^{-1} F_1(t+1)\big(F_1(t)+F_2(t)\big)^{-1} F_2(t) x.$$

   Since this must hold for all x, and since we require that $F_i(t+1)$ be symmetric, it must be the case that

   $$F_1(t+1) = \frac{1}{2}\Big[G_1 F_2(t)^{-1}\big(F_1(t) + F_2(t)\big) + \big(F_1(t) + F_2(t)\big) F_2(t)^{-1} G_1\Big]. \tag{10}$$

   Similarly, for CCP 2,

   $$F_2(t+1) = \frac{1}{2}\Big[G_2 F_1(t)^{-1}\big(F_1(t) + F_2(t)\big) + \big(F_1(t) + F_2(t)\big) F_1(t)^{-1} G_2\Big]. \tag{11}$$

First, consider the scalar, single-instrument case ($m = 1$). Suppose the CCPs disagree in their price impact beliefs and, without loss of generality, $G_1 > G_2$, so CCP 1 believes the price impact is greater than CCP 2 does. Then, for $t \geq 1$, the best response dynamics yield

$$\frac{F_2(t)}{F_1(t)} = \frac{G_2}{G_2}\frac{F_2(t-1)}{F_1(t-1)} = \left(\frac{G_2}{G_1}\right)^{t+1},$$

where the first equality follows from (10)–(11) and the second equality follows by induction. As

$t \to \infty$, we have that $F_2(t)/F_1(t) \to 0$, and this implies that

$$x_1(t) = \big(1 + F_1(t)/F_2(t)\big)^{-1} x \to 0, \quad x_2(t) = \big(1 + F_2(t)/F_1(t)\big)^{-1} x \to x.$$

In other words, asymptotically, CCP 2 clears a larger fraction of the position by charging lower margin. Due to the convexity of the quadratic total margin function, this forces CCP 1 to charge increasingly higher margins in order to cover liquidation costs. Asymptotically, CCP 1 has an infinite initial margin and is thus driven out of the clearing market. We call this a "race to the bottom" because the CCP with the lower price impact ultimately determines margin costs for the entire market.

More generally, we can expand our discussion above to the multidimensional case:

Proposition 2. *Suppose that the CCPs differ in their price impact belief matrices $G_1, G_2 \in \mathbb{R}^{m \times m}$. Then:*

(i) the matrices $\big(F_1(t+1), F_2(t+1)\big)$ defined in (10)–(11) are positive definite for all $t \geq 0$,

(ii) if the spectral radius of $G_1^{-1} G_2$ is strictly less than 1, as $t \to \infty$,

$$F_2(t) F_1(t)^{-1} \to 0, \quad x_1(t) \to 0, \quad x_2(t) \to x.$$

Part (i) shows that the best response dynamics suggested earlier are well-defined for all $t \geq 0$. Part (ii) states that, if the price impact beliefs of CCP 2 are "smaller" (in the sense of the spectral radius of their ratio) than those of CCP 1, CCP 1 will ultimately be driven out of the clearing market. If $G_1 \succ G_2$ in the positive definite ordering, i.e., if the margin required by the matrix G_1 dominates that of G_2 for *every* portfolio, then the spectral radius of $G_1^{-1} G_2$ must be less than 1 and part (ii) applies.

5.3. Partitioned Clearing

Thus far, we have assumed that both CCPs clear the entire universe of available instruments. But the first decision a CCP makes is which types of instruments to clear. We now extend Theorem 1 by expanding the strategy space for each CCP to include the choice of instruments to clear as well as the initial margin to charge. We continue to suppose that each CCP's belief about true price impact is given by a symmetric, positive definite matrix $G_i \in \mathbb{R}^{m \times m}$, where m is the total number of securities available for clearing.

We assume that a CCP clears all linear combinations of the securities it clears, and does not clear linear combinations that include securities that it does not clear. So, the choice of a subset of security types is a choice of subspace of $\mathbb{R}^m$. Write $m = m_1 + m_2 + m_3$, where[5]

$$\begin{aligned}
m_1 &= \text{number of security types cleared only by CCP 1,}\\
m_2 &= \text{number of security types cleared by both CCPs,}\\
m_3 &= \text{number of security types cleared only by CCP 2.}
\end{aligned}$$

We also assume that the security types are numbered in this order, so that the first m_1 types are cleared only by CCP 1, and so on.

The margin matrices F_1 and F_2 have dimensions $(m_1 + m_2) \times (m_1 + m_2)$ and $(m_2 + m_3) \times (m_2 + m_3)$, respectively. Denote by $P_1 \in \mathbb{R}^{(m_1+m_2)\times m}$ the matrix of the projection of $\mathbb{R}^m$ onto the first $m_1 + m_2$ coordinates corresponding to swap types cleared by CCP 1. Similarly, denote by $P_2 \in \mathbb{R}^{(m_2+m_3)\times m}$ the matrix of the projection onto the last $m_2 + m_3$ coordinates corresponding to swap types cleared by CCP 2. Finally, let the notation $\mathbf{0}_k \in \mathbb{R}^k$ denote a zero row vector of length k, and the notation $(x_1^\top, \mathbf{0}_{m_3})$ and $(\mathbf{0}_{m_1}, x_2^\top)$ denote the lifting of vectors $x_1 \in \mathbb{R}^{m_1+m_2}$ and $x_2 \in \mathbb{R}^{m_2+m_3}$ from the subspaces cleared by the two CCPs to full-length portfolio vectors.

With the above notation in place, we can make the following definition:

Definition 3 (Partitioned Equilibrium with Linear Price Impact)**.** *Given price impact belief matrices $G_1, G_2 \in \mathbb{R}^m$ for the two CCPs, a partitioned equilibrium is defined by*

1. *a partition (m_1, m_2, m_3) of the m swap types,*
2. *allocation functions $x_1\colon \mathbb{R}^m \to \mathbb{R}^{m_1+m_2}$ and $x_2\colon \mathbb{R}^m \to \mathbb{R}^{m_2+m_3}$,*
3. *price impact margin matrices $F_1 \in \mathbb{R}^{m_1+m_2}$, $F_2 \in \mathbb{R}^{m_2+m_3}$,*

satisfying, for each portfolio $x \in \mathbb{R}^m$,

1. *$(x_1(x), x_2(x))$ is an optimal solution to the clearing member's optimization problem*

$$\underset{x_1\in\mathbb{R}^{m_1+m_2},\ x_2\in\mathbb{R}^{m_2+m_3}}{\text{minimize}} \left\{ x_1^\top F_1 x_1 + x_2^\top F_2 x_2 \;\middle|\; \textit{subject to } (x_1^\top, \mathbf{0}_{m_3}) + (\mathbf{0}_{m_1}, x_2^\top) = x \right\}, \tag{12}$$

2. *each CCP i collects liquidity margin based on its true price impact beliefs, i.e.,*

$$x_1(x)^\top F_1 x_1(x) = x_1(x)^\top P_1 G_1 x, \quad x_2(x)^\top F_2 x_2(x) = x_2(x)^\top P_2 G_2 x. \tag{13}$$

The following theorem characterizes partitioned equilibria:

Theorem 2. *A necessary and sufficient condition for a partitioned equilibrium with linear price impact is that the price impact belief matrices G_1, G_2 have a common block diagonal structure*

$$G_i = \begin{pmatrix} G_i(1,1) & & \\ & G_i(2,2) & \\ & & G_i(3,3) \end{pmatrix}, \quad i \in \{1,2\}, \tag{14}$$

[5] Without loss of generality, securities cleared by neither CCP can be excluded from consideration.

with $G_i(1,1) \in \mathbb{R}^{m_1 \times m_1}$*,* $G_i(2,2) \in \mathbb{R}^{m_2 \times m_2}$ *,*$G_i(3,3) \in \mathbb{R}^{m_3 \times m_3}$*, where the submatrices satisfy*

$$G_1(2,2) = G_2(2,2) \triangleq G(2,2). \tag{15}$$

In this case, CCP 1 clears the first $m_1 + m_2$ *swap types, CCP 2 clears the last* $m_2 + m_3$ *swap types, and they choose margin matrices*

$$F_1 = \begin{pmatrix} G_1(1,1) & \\ & F_1(2,2) \end{pmatrix}, \quad F_2 = \begin{pmatrix} F_2(2,2) & \\ & G_2(3,3) \end{pmatrix}, \tag{16}$$

for any symmetric, positive definite matrices $F_1(2,2), F_2(2,2) \in \mathbb{R}^{m_2 \times m_2}$ *satisfying*

$$F_1(2,2)^{-1} + F_2(2,2)^{-1} = G(2,2)^{-1}. \tag{17}$$

Theorem 2 establishes a number of requirements for partitioned equilibria. Condition (15) implies that the two CCPs need to have common beliefs on price impact for the instruments they both clear. The block structure requirement in (14) implies that an instruments cleared by only a single CCP cannot have any cross-price impact with any swap clear by the other CCP.

Next, we consider a refinement of the partitioned equilibrium of Definition 3:

Definition 4 (Stable Equilibrium)**.** *A partitioned equilibrium* $(m_1, m_2, m_3, F_1, F_2, x_1, x_2)$ *is called stable if it is undominated, in the sense that there exists no other equilibrium* $(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, \tilde{F}_1, \tilde{F}_2, \tilde{x}_1, \tilde{x}_2)$ *such that*

$$x_1(x)^\top F_1 x_1(x) + x_2(x)^\top F_2 x_2(x) \geq \tilde{x}_1(x)^\top \tilde{F}_1 \tilde{x}_1(x) + \tilde{x}_2(x)^\top \tilde{F}_2 \tilde{x}_2(x), \quad \textit{for all } x \in \mathbb{R}^m,$$

and that the inequality holds strictly for some $x \in \mathbb{R}^m$*.*

An equilibrium with the block structure (14)–(15) may fail to be stable in the following way: Suppose that among the first m_1 instruments (those cleared only by CCP 1) there is some instrument with index j for which $G_2(j,j) < G_1(j,j)$, and suppose that $G_1(j,k) = G_2(j,k) = 0$, for all $k \neq j$. Then we can construct another equilibrium by moving instrument j from the set cleared only by CCP 1 to the set cleared only by CCP 2 and reduce the total margin charged.

The following result provides a sufficient condition for stability:

Proposition 3 (Stable Partitioned Equilibrium)**.** *A partitioned equilibrium is stable if*

$$G_1(1,1) \preceq G_2(1,1), \quad G_1(3,3) \succeq G_2(3,3), \tag{18}$$

in the positive definite order.

Proposition 3 states that an equilibrium is stable if each CCP collects less margin for the set of instruments it clears exclusively than the other CCP would. For example, if $G_1 \succ G_2$, then having CCP 2 clear all positions alone is the unique stable equilibrium.

6. Adding Uncertainty

To this point, we have assumed a completely deterministic model in which each CCP is able to infer a clearing member's full portfolio vector x from the portion cleared by that CCP by effectively inverting the solution to the clearing member's problem (5). In this section, we extend our results by adding *uncertainty.* We consider two forms of uncertainty: uncertainty in the CCPs' inferences about the clearing member's portfolio, and uncertainty in the CCPs' beliefs about the true price impact.

To incorporate uncertainty in the CCPs' beliefs, we take the price impact matrices G_1 and G_2 to be stochastic. We assume that these matrices are almost surely symmetric and positive definite. The same is then true of their expectations $\mathsf{E}[G_i]$, $i \in \{1, 2\}$.

We use a simple model of the CCP's uncertainty about the clearing member's portfolio. We suppose that when CCP i clears a portion x_i of the full portfolio x, it forms an estimate

$$\hat{x}_i = x + \epsilon_i,$$

of the full portfolio, with $\mathsf{E}[\epsilon_i] = 0$, $i \in \{1, 2\}$. In other words, a CCP cannot perfectly infer the clearing member's full portfolio, but it can form an unbiased estimate $\hat{x}_i$ of the full portfolio.

This model provides a reduced-form description of the many sources of uncertainty that would in practice prevent a CCP from reverse engineering a clearing member's portfolio. In particular, a CCP may not have perfect information about its competitors' margin functions, and considerations other than margin minimization may influence the clearing member's allocation. Our key assumption is that these factors do not lead the CCP to systematically misjudge the clearing member's full portfolio.

To extend our earlier results to include uncertainty, we suppose that each CCP sets its margin function to collect sufficient margin in expectation. More precisely, we define an equilibrium as in Definition 2 but replacing the last condition given there with the following condition:

$$x_i^\top F_i(x_i) = \mathsf{E}\Big[x_i^\top G_i(\hat{x}_i)\Big], \quad i \in \{1, 2\}. \tag{19}$$

Proposition 4. *Suppose that for each CCP i, ϵ_i and G_i are uncorrelated. Then a necessary and sufficient condition for equilibrium with linear price impact is that the two CCPs have common views on the mean market impact, i.e., that $\mathsf{E}[G_1] = \mathsf{E}[G_2] \triangleq G$.*

In this case, all equilibria are determined by the symmetric, positive definite solutions $F_1, F_2 \in \mathbb{R}^{m\times m}$ to the equation

$$G^{-1} = F_1^{-1} + F_2^{-1}.$$

Proof. Because G_i is uncorrelated with ϵ_i, we have

$$\mathsf{E}\Big[x_i^\top G_i(\hat{x}_i)\Big] = \mathsf{E}\Big[x_i^\top G_i(x + \epsilon_i)\Big] = x_i^\top \mathsf{E}[G_i](x + \mathsf{E}[\epsilon_i]) = x_i^\top \mathsf{E}[G_i]x.$$

Thus, (19) reduces to $x_i^\top F_i(x_i) = x_i^\top Gx$. The result now follows from Theorem 1. ■

7. A Single Instrument with General Price Impact

In general, it is not easy to solve for equilibrium under nonlinear price impact models. It is, however, possible to characterize the scalar case. In this section, we specialize to the case of a single instrument ($m = 1$) in which the portfolio $x \in \mathbb{R}$ is scalar. Each CCP i has price impact belief $G_i(x)$ and margin function $f_i(x) = xF_i(x)$.

Suppose that (F_1, F_2, x_1, x_2) form an equilibrium according to Definition 2. Then, first order necessary and sufficient conditions for the clearing member's problem (5) are that

$$F_1(x_1) + x_1 F_1'(x_1) = F_2(x_2) + x_2 F_2'(x_2). \tag{20}$$

Also, the sufficient margin condition is equivalent to

$$F_i(x_i) = G_i(x). \tag{21}$$

In the following, we use

$$f^*(x) \triangleq \sup_{y\in\mathbb{R}} \{xy - f(y)\}$$

to denote the convex conjugate of a function of f on $\mathbb{R}$.

Theorem 3. *(i) If the CCPs have common beliefs $G_1 = G_2 \triangleq G$, then an equilibrium exists. All equilibria result in proportional allocations $x_1 = \alpha x$ and $x_2 = (1-\alpha)x$, for some $\alpha \in (0,1)$, and*

$$F_1(x) = G(x/\alpha), \quad F_2(x) = G(x/(1-\alpha)).$$

(ii) If an equilibrium with proportional allocations exists, then the CCPs have common beliefs $G_1 = G_2$.

(iii) In any equilibrium with common beliefs, $f_{\mathsf{eff}}(x) = g(x) \triangleq xG(x)$, meaning that the effective margin equals the shared view on the required margin. Moreover, the common belief can be recovered from the individual margin functions through the relation

$$g = (f_1^* + f_2^*)^*. \tag{22}$$

Proof. (i) For the existence of an equilibrium, it suffices to show that

$$x_1 = x_2 = x/2, \quad F_1(x) = F_2(x) = G(2x),$$

is an equilibrium. This follows from the fact that (20) and (21) hold in this case.

Next, we establish that all equilibria result in proportional allocations. If $G_1 = G_2 \triangleq G$, (21) implies $F_1(x_1) = F_2(x_2)$, so (20) implies

$$x_1 F_1'(x_1) = x_2 F_2'(x_2). \tag{23}$$

Differentiating (21) with respect to x, we get that

$$F_i'(x_i)x_i' = G_i'(x).$$

This yields

$$F_1'(x_1)x_1' = F_2'(x_2)x_2'. \tag{24}$$

This implies that x_1 and x_2 are strictly increasing and therefore strictly positive for $x > 0$. For $x > 0$, combining the (23) and (24), we get

$$\frac{x_1'}{x_1} = \frac{x_2'}{x_2}.$$

So $x_2 = cx_1$ for some constant $c > 0$, and the claim holds with $\alpha \triangleq 1/(1+c)$.

(ii) Suppose $x_1 = \alpha x$ and $x_2 = (1-\alpha)x$, and define

$$h(x) \triangleq F_1(x_1) - F_2(x_2) = F_1(\alpha x) - F_2((1-\alpha)x).$$

Differentiating this with respect to x, we have

$$h'(x) = \alpha F_1(\alpha x) - (1-\alpha)F_2'((1-\alpha)x).$$

But using the first-order condition (20), we can write h as

$$h(x) = -x_1 F_1'(x_1) + x_2 F_2'(x_2) = -\alpha x F_1(\alpha x) + (1-\alpha)x F_2'((1-\alpha)x) = -xh'(x).$$

Then, $h(x) + xh'(x) = 0$, which means that $xh(x)$ is a constant, so we must have $h(x) \equiv 0$. In other words, $F_1(x_1) = F_2(x_2)$, and thus $G_1 = G_2$ by (21).

(iii) We take the conjugate of the effective margin f_{eff} in (5). Because f_i is convex and continuous, we have, by Rockafellar (1997, Theorem 16.4),

$$f_{\mathsf{eff}}^* = (f_1 \square f_2)^* = f_1^* + f_2^*.$$

The infimal convolution of convex, continuous functions is also convex and continuous so

$$f_{\mathsf{eff}} = f_{\mathsf{eff}}^{**} = (f_1^* + f_2^*)^*,$$

using Theorem 12.2 and Corollary 12.2.1 of Rockafellar (1997). Now, notice that in equilibrium we always have

$$f_1(x_1) + f_2(x_2) = x_1 F_1(x_1) + x_2 F_2(x_2) = x_1 G_1(x) + x_2 G_2(x) = xG(x).$$

Then by the definition of infimal convolution, we have $g(x) = xG(x) = f_{\mathsf{eff}}(x)$. ■

In the case of linear price impact, the total margin functions f_1, f_2 are quadratic, and (22) leads to

$$g^*(x) = G^{-1}x^2 = f^*_{\mathsf{eff}}(x) = F_1^{-1}x^2 + F_2^{-1}x^2,$$

for all $x \in R$, so that

$$G^{-1} = F_1^{-1} + F_2^{-1}. \tag{25}$$

This is just the scalar case of Theorem 1.

As another example, suppose the price impact function takes the form $G(x) \triangleq cx^\beta$, given an exponent $\beta > 0$. Theorem 3 yields an equilibrium with $F_i(x) \triangleq b_i x^\beta$, $i \in \{1, 2\}$ so long as

$$b_1^{-1/\beta} + b_2^{-1/\beta} = c^{-1/\beta}. \tag{26}$$

To see this, first notice that $g(x) = cx^{\beta+1}$, hence

$$g^*(y) = c^{-1/\beta} \frac{x^{1+1/\beta}(\beta+1)^{-1/\beta}}{1 + 1/\beta}.$$

Similarly,

$$f_i^*(y) = b_i^{-1/\beta} \frac{x^{1+1/\beta}(\beta+1)^{-1/\beta}}{1 + 1/\beta}.$$

Then (26) is just a result of applying (22). Note that (25) is a special case of (26) with $\beta = 1$.

Theorem 3 leaves open the possibility of an equilibrium in which the CCPs have different views, which would require that the allocations x_1, x_2 not be proportional.

8. Implications and Concluding Remarks

Our analysis has relied on simplifying assumptions and a stylized model of the complex decisions faced by central counterparties and their clearing members. Nevertheless, this analysis has practical implications for the functioning of derivatives markets.

- A CCP's initial margin requirements should reflect liquidity costs as well as market risk. Liquidity costs increase more than proportionally with position size, so margin requirements should as well. This is a premise of our analysis but it bears repeating. In responding to comments on its proposed rules, the CFTC specifically declined recommendations requiring that position concentration be factored into margin calculations, leaving the matter to the discretion of each CCP; (see Commodity Futures Trading Commission, 2011, p. 69366).

- When incorporating liquidity costs into margin requirements, a CCP also needs to consider a clearing member's positions at other CCPs. If the clearing member defaults, its positions at all CCPs will hit the market simultaneously, so price impact is determined by the clearing member's combined positions, not its position at a single CCP. Moreover, superlinear margin charges designed to capture liquidity costs create an incentive for clearing members to split positions across CCPs, thus amplifying the effect of hidden illiquidity.

- To counteract this effect, CCPs and clearing members need to share information about positions across CCPs. This is difficult to achieve, given the sensitivity of the information. One approach would be for each CCP to make a conservative assumption about a clearing member's positions at other CCPs (with a correspondingly conservative margin charge) and create a positive incentive for clearing members to provide this information by offering a potential margin reduction in exchange. A CCP could make a conservative assumption by comparing the positions in a contract it clears with the total outstanding positions in that contract across all participants and CCPs. This type of aggregate data is collected by swap data repositories, as mandated by the Dodd-Frank Act.

- Our analysis also points to the need for CCPs to share information about liquidation costs. The relevant costs would be incurred at the failure of a major swaps dealer and are not easily gleaned from historical data. To better estimate price impacts, CCPs could require their clearing members to regularly provide prices and quantities at which they are committed to buy or sell upon the default of another member.

- A CCP is required to test its default management process, through which a defaulting member's positions are unwound, at least annually. These default management drills should explicitly account for the actions of other CCPs directly affected by the same member's default.

- Market participants and regulators have recently called for standardized stress tests for CCPs. Our analysis points to the need for each CCP's stress scenarios to include the actions of other CCPs. This would be in contrast to the current regulatory stress tests for banks, which treat each bank in isolation.

These recommendations are not necessarily easy to implement. Each of these steps requires further research.

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A. Theorem Proofs

A.1. Proof of Theorem 1

We will make frequent use of the fact that our definitions require the matrices F_i and G_i to be symmetric and positive definite.

Necessity. Suppose (x_1, x_2, F_1, F_2) defines an equilibrium. The first-order conditions for the clearing member's optimization problem (9) yield

$$x_1 = (F_1 + F_2)^{-1} F_2 x. \tag{27}$$

The sufficient margin condition for CCP 1 implies

$$x_1^\top G_1 x = x_1^\top F_1 x_1,$$

for all x. We can use (27) to write this as

$$x_1^\top G_1 (F_1 + F_2) F_2^{-1} x_1 = x_1^\top F_1 x_1. \tag{28}$$

We need this to hold for all $x_1 \in \mathbb{R}^m$ because from (27) we see that x_1 ranges over all of $\mathbb{R}^m$ as x does. Thus, the matrices on the two sides of (28) must have the same symmetric parts. Applying the same argument to CCP 2, this yields

$$\begin{aligned} F_1 &= \frac{1}{2}(G_1 F_2^{-1} F_1 + F_1 F_2^{-1} G_1) + G_1 & (29) \\ F_2 &= \frac{1}{2}(G_2 F_1^{-1} F_2 + F_2 F_1^{-1} G_2) + G_2. & (30) \end{aligned}$$

We can rewrite these equations as

$$F_1 = \frac{1}{2}(I + F_1F_2^{-1})G_1 + \frac{1}{2}G_1(I + F_2^{-1}F_1) \tag{31}$$
$$F_2 = \frac{1}{2}(I + F_2F_1^{-1})G_2 + \frac{1}{2}G_2(I + F_1^{-1}F_2). \tag{32}$$

Each of these equations has the form

$$B = AX + X^\top A^\top$$

According to Braden (1998, Theorem 1), the solutions to (31) and (32) take the following form: for some skew-symmetric [6] matrices Q_1, Q_2,

$$G_1 = (I + F_1F_2^{-1})^{-1}F_1 + \frac{1}{2}Q_1(I + F_1F_2^{-1})$$
$$G_2 = (I + F_2F_1^{-1})^{-1}F_2 + \frac{1}{2}Q_2(I + F_2F_1^{-1}).$$

Making the substitutions

$$(I + F_1F_2^{-1})^{-1} = F_2(F_2 + F_1)^{-1}, \quad (I + F_2F_1^{-1})^{-1} = F_1(F_2 + F_1)^{-1},$$

we get

$$G_1 = F_2(F_2 + F_1)^{-1}F_1 + \frac{1}{2}Q_1(I + F_1F_2^{-1}) \tag{33}$$
$$G_2 = F_1(F_2 + F_1)^{-1}F_2 + \frac{1}{2}Q_2(I + F_2F_1^{-1}). \tag{34}$$

Next observe that for any symmetric, invertible A, B,

$$\begin{aligned} A(A+B)^{-1}B &= A(I + B^{-1}A)^{-1} \\ &= A[A^{-1}(A^{-1} + B^{-1})^{-1}] \\ &= (A^{-1} + B^{-1})^{-1}. \end{aligned}$$

Thus, we can write (33)–(34) as

$$G_1 = (F_1^{-1} + F_2^{-1})^{-1} + \frac{1}{2}Q_1(I + F_1F_2^{-1}) \tag{35}$$
$$G_2 = (F_1^{-1} + F_2^{-1})^{-1} + \frac{1}{2}Q_2(I + F_2F_1^{-1}). \tag{36}$$

We will show that $Q_1 = Q_2 = 0$. It will then follow that

$$G_1 = (F_1^{-1} + F_2^{-1})^{-1} = G_2 \triangleq G$$

[6] A square matrix A is skew-symmetric if it satisfies the condition $-A = A^\top$.

and therefore

$$G^{-1} = F_1^{-1} + F_2^{-1}. \tag{37}$$

It remains to show that $Q_1 = Q_2 = 0$. Observe that the first term on the right side of (35) and (36) is symmetric, so the last term must be symmetric as well. Also, because the F_i are positive definite, $F_1 F_2^{-1}$ and $F_2 F_1^{-1}$ have positive eigenvalues (even though they are not necessarily positive definite). Thus, it suffices to show that if Q is skew-symmetric and $X = F_1 F_2^{-1}$ has positive eigenvalues, then $Q(I + X)$ cannot be symmetric unless $Q = 0$.

If $Q(I + X)$ is symmetric, $Q + QX = -Q + X^\top Q^\top$ and

$$2Q = (X^\top Q^\top - QX). \tag{38}$$

Any skew-symmetric matrix Q can be written in the form $Q = U\Lambda U^\top$, where U is orthogonal, and

$$\Lambda = \begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \lambda_{\frac{m-k}{2}} & \\ & & & -\lambda_{\frac{m-k}{2}} & 0 & \\ & & & & & \mathbf{0}_{k\times k} \end{pmatrix},$$

where $\mathbf{0}_{k\times k}$ is a block of zeros, for some k. We always have $m - k$ even, and k may be zero if m is even. We can write (38) as

$$2U\Lambda U^\top = (X^\top U \Lambda^\top U^\top - U\Lambda U^\top X)$$

and then

$$2\Lambda = (U^\top X^\top U \Lambda^\top - \Lambda U^\top X U) = (\tilde{X}^\top \Lambda^\top - \Lambda \tilde{X}),$$

where $\tilde{X}$ has the same eigenvalues as X. So, it suffices to consider (38) in the case $Q = \Lambda$,

$$2\Lambda = (X^\top \Lambda^\top - \Lambda X). \tag{39}$$

With Λ as given above, we claim that X must have a block decomposition

$$X = \begin{pmatrix} A & \mathbf{0}_{m-k\times k} \\ C & B \end{pmatrix}. \tag{40}$$

If $k = 0$, there is nothing to prove, so suppose $k \geq 1$. Consider any X_{ij} with $i \leq m - k$ and

$j > m-k$. Denote by $\Lambda_{\ell i}$ the unique nonzero entry in the ith column of Λ. Then if (39) holds,

$$\begin{aligned} 0 = 2\Lambda_{\ell j} &= (\Lambda X)_{j\ell} - (\Lambda X)_{\ell j} \\ &= \sum_m \Lambda_{jm} X_{m\ell} - \Lambda_{\ell i} X_{ij} \\ &= -\Lambda_{\ell i} X_{ij}, \end{aligned}$$

so $X_{ij} = 0$, which confirms (40). As a consequence of (39) and (40), we have

$$\begin{aligned} 2\lambda_1 = 2\Lambda_{12} &= (\Lambda A)_{21} - (\Lambda A)_{12} \\ &= -\lambda_1 A_{11} - \lambda_1 A_{22} \end{aligned}$$

so $A_{11} + A_{22} = -2$. The same calculation applies for all $\lambda_2, \ldots, \lambda_{(m-k)/2}$, so the trace of A is negative (in fact, equal to $-(m-k)$), so A must have at least one negative eigenvalue. But from (40) we see that every eigenvalue of A is an eigenvalue of X, and we know that X has only positive eigenvalues. We conclude that the only solution to (39) is $\Lambda = 0$.

Sufficiency. Suppose the CCPs have common views on market impact $G_1 = G_2 = G$, and suppose F_1, F_2 satisfy (8). Then (27) and (28) hold, and F_1, F_2 define an equilibrium. $\square$

A.2. Proof of Proposition 2

In order to establish part (i), we will prove the following by induction: for all times $t \geq 0$,

$$F_i(t) \succ 0, \quad i \in \{1, 2\}, \tag{41}$$

$$F_1(t) F_2(t)^{-1} = \left(G_1 G_2^{-1}\right)^{t+1}. \tag{42}$$

Clearly (41)–(42) hold when $t = 0$.

Suppose they hold for t. Then, substituting (42) in (10)–(11),

$$\begin{aligned} F_1(t+1) &= \frac{1}{2}\left[G_1\left(G_2^{-1}G_1\right)^{t+1} + 2G_1 + \left(G_1 G_2^{-1}\right)^{t+1} G_1\right] = G_1 + G_1\left(G_2^{-1}G_1\right)^{t+1}, \\ F_2(t+1) &= \frac{1}{2}\left[G_2\left(G_1^{-1}G_2\right)^{t+1} + 2G_2 + \left(G_2 G_1^{-1}\right)^{t+1} G_2\right] = G_2 + G_2\left(G_1^{-1}G_2\right)^{t+1}. \end{aligned}$$

Then, since $G_1, G_2 \succ 0$, clearly (41) holds at time $t+1$. Further,

$$\begin{aligned} F_1(t+1)F_2(t+1)^{-1} &= G_1\left[I + \left(G_2^{-1}G_1\right)^{t+1}\right]\left[I + \left(G_1^{-1}G_2\right)^{t+1}\right]^{-1} G_2^{-1} \\ &= G_1\left(G_2^{-1}G_1\right)^{t+1} G_2^{-1} \\ &= \left(G_1 G_2^{-1}\right)^{t+2}, \end{aligned}$$

establishing (42) at time $t+1$.

For part (ii), since the spectral radius of $G_1^{-1}G_2$ is less than 1,

$$\lim_{t\to\infty}\left(G_1^{-1}G_2\right)^t = 0.$$

This implies that

$$\begin{aligned}
\lim_{t\to\infty} x_2(t) &= \lim_{t\to\infty}\left(F_1(t)+F_2(t)\right)^{-1}F_1(t)x \\
&= \lim_{t\to\infty}\left[I + F_1(t)^{-1}F_2(t)\right]^{-1} x \\
&= \lim_{t\to\infty}\left[I + (G_1^{-1}G_2)^t\right]^{-1} x \\
&= x,
\end{aligned}$$

Further,

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x - x_2(t) = 0.$$

□

A.3. Proof of Theorem 2

Sufficiency. Let the number of rows (and columns) in the three blocks be m_1, m_2, and m_3. We claim that we get an equilibrium if CCP 1 clears the first m_1+m_2 security types, CCP 2 clears the last m_2+m_3 security types, and they choose margin matrices

$$F_1 = \begin{pmatrix} G_1(1,1) & \\ & F_1(2,2) \end{pmatrix}, \quad F_2 = \begin{pmatrix} F_2(2,2) & \\ & G_2(3,3) \end{pmatrix}, \tag{43}$$

for some symmetric $F_1(2,2)$, $F_2(2,2)$ satisfying

$$F_1(2,2)^{-1} + F_2(2,2)^{-1} = G(2,2)^{-1}. \tag{44}$$

To show that this holds, for any $x \in \mathbb{R}^m$ we can write

$$x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad x_1 = \begin{pmatrix} u \\ v_1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} v - v_1 \\ w \end{pmatrix},$$

$u \in \mathbb{R}^{m_1}$, $v, v_1 \in \mathbb{R}^{m_2}$, and $w \in \mathbb{R}^{m_3}$. The minimization over (x_1, x_2) in (12) reduces to a minimization over v_1 with solution

$$v_1 = (F_1(2,2) + F_2(2,2))^{-1}F_2(2,2)v.$$

To verify the first condition in (13) observe that

$$x_1^\top F_1 x_1 = u^\top G_1(1,1)u + v_1^\top F_1(2,2)v_1 \tag{45}$$

and

$$x_1^\top P_1 G_1 x = u^\top G_1(1,1)u + v_1^\top G(2,2)v. \tag{46}$$

But (44) implies that

$$G(2,2) = (F_1^{-1}(2,2) + F_2^{-1}(2,2))^{-1} = F_1(2,2)(F_1(2,2) + F_2(2,2))^{-1} F_2(2,2)$$

so (45) and (46) are equal. A similar argument verifies the second condition in (13).

Necessity. We now show that if (G_1, G_2) admit an equilibrium $(F_1, F_2, m_1, m_2, m_3)$, then (G_1, G_2) have the block structure in (14)–(15).

First consider any securities i and j cleared only by CCPs 1 and 2, respectively. Write $c_1(i,j)$ for the (i,j) entry of F_1, and write $\bar{c}_1(i,j)$, $\bar{c}_1(i,i)$ for the corresponding entries of G_1. Consider a portfolio holding u units of i and w units of j. Condition (13) requires

$$u^2 c_1(i,i) = u(\bar{c}_1(i,i)u + \bar{c}_1(i,j)w)$$

for all u and w. The case $w = 0$ implies that $c_1(i,i) = \bar{c}_1(i,i)$, and then any $w \neq 0$ implies $\bar{c}_1(i,j) = 0$. Thus, the block $G_1(1,3) = G_1(3,1)$ is identically zero. By the same argument, $G_2(1,3) = G_2(3,1) = 0$.

Now suppose security j is cleared by both CCPs and consider a portfolio holding u units of i and $v \neq 0$ units of j, with v_1 units cleared through CCP 1 and $v - v_1$ units cleared through CCP 2. To solve (12), the clearing member chooses v_1 to minimize

$$u^2 c_1(i,i) + 2uv_1 c_1(i,j) + c_1(j,j)v_1^2 + c_2(j,j)(v - v_1)^2,$$

which yields

$$v_1 = \frac{c_2(j,j)v - c_1(i,j)u}{c_1(j,j) + c_2(j,j)}. \tag{47}$$

To satisfy (13), we need to have

$$u^2 c_1(i,i) + 2uv_1 c_1(i,j) + c_1(j,j)v_1^2 = u^2 \bar{c}_1(i,i) + u(v_1 + v)\bar{c}_1(i,j) + vv_1 \bar{c}_1(j,j).$$

We have already established that $c_1(i,i) = \bar{c}_1(i,i)$, so this entails

$$c_1(j,j)\frac{v_1^2}{v^2} - \bar{c}_1(j,j)\frac{v_1}{v} = \bar{c}_1(i,j)u\left[\frac{1}{v} + \frac{v_1}{v^2}\right] - 2c_1(i,j)u\frac{v_1}{v^2}. \tag{48}$$

If neither $c_1(i,j)$ nor $\bar{c}_1(i,j)$ is zero, then $v_1 = 0$ in (47) at some $u \neq 0$ but not in (48). So, suppose $c_1(i,j) = 0$. Then v_1/v in (47) is a constant, independent of u. But for the same to hold in (48) we must have $\bar{c}_1(i,j) = 0$. We conclude that $G_1(1,2) = 0$, and the same argument shows $G_1(3,2) = 0$.
□

A.4. Proof of Proposition 3

First write $\{G_1, G_2\}$ in the same block diagonal structure with k as large as possible, such that

$$G_i = \begin{pmatrix} G_i(1,1) & & & \\ & G_i(2,2) & & \\ & & \ddots & \\ & & & G_i(k,k) \end{pmatrix}$$

where $G_i(j,j) \in \mathbb{R}^{m_j \times m_j}$, $\sum_{j=1}^{k} m_j = m$, and for $\mathcal{B} \cup \mathcal{F}_1 \cup \mathcal{F}_2 = \{1, 2, ..., k\}$ the following hold:

1. for $j \in \mathcal{B}$, $G_1(j,j) = G_2(j,j)$
2. for $j \in \mathcal{F}_1$, $G_2(j,j) \succeq G_1(j,j)$ and $G_1(j,j) \neq G_2(j,j)$
3. for $j \in \mathcal{F}_2$, $G_1(j,j) \succeq G_2(j,j)$ and $G_1(j,j) \neq G_2(j,j)$

This means that the two CCPs disagree for security classes in $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_2$ and agree on security classes in $\mathcal{B}$. There are no cross impacts between securities in different security classes.

Let E_1 denote an equilibrium in Definition 3. From Theorem 2, we know that in any partitioned equilibrium, CCPs can only jointly clear security classes for which they have the same market beliefs. For equilibrium E_1, we assume that CCP 1 clears security classes in $\mathcal{S}_1$, and CCP 2 clears security classes in $\mathcal{S}_2$. Then we have $\mathcal{F} \cap \mathcal{S}_1 = \mathcal{F}_1$, $\mathcal{F} \cap \mathcal{S}_2 = \mathcal{F}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq \mathcal{B}$.

For a partitioned equilibrium E_2 other than E_1, we assume that CCP 1 clears security classes in $\tilde{\mathcal{S}}_1$, and CCP 2 clears security classes in $\tilde{\mathcal{S}}_2$. We have:

$$\tilde{\mathcal{S}}_1 \cap \tilde{\mathcal{S}}_2 \subseteq \mathcal{B}, \quad \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2 = \{1, 2, ..., k\}$$

For any position $x^\top = (x^\top(1), .., x^\top(k))$, with $x(j) \in \mathbb{R}^{m_j}$, by definition the total margin collected in equilibrium E_1 is

$$x_1^\top F_1 x_1 + x_2^\top F_2 x_2 = \sum_{j \in \mathcal{S}_1 \cap \mathcal{F}} x^\top(j) G_1(j,j) x(j) + \sum_{j \in \mathcal{S}_2 \cap \mathcal{F}} x^\top(j) G_2(j,j) x(j) + \sum_{j \in \mathcal{B}} x^\top(j) G_1(j,j) x(j) \tag{49}$$

The total margin collected in equilibrium E_2 $(\tilde{x}_1, \tilde{x}_2, \tilde{F}_1, \tilde{F}_2)$ is

$$\tilde{x}_1^\top \tilde{F}_1 \tilde{x}_1 + \tilde{x}_2^\top \tilde{F}_2 \tilde{x}_2 = \sum_{j \in \tilde{\mathcal{S}}_1 \cap \mathcal{F}} x^\top(j) G_1(j,j) x(j) + \sum_{j \in \tilde{\mathcal{S}}_2 \cap \mathcal{F}} x^\top(j) G_2(j,j) x(j) + \sum_{j \in \mathcal{B}} x^\top(j) G_1(j,j) x(j) \tag{50}$$

Taking the difference between (49) and (50), we get

$$
\begin{aligned}
& x_1^\top F_1 x_1 + x_2^\top F_2 x_2 - \tilde{x}_1^\top \tilde{F}_1 \tilde{x}_1 - \tilde{x}_2^\top \tilde{F}_2 \tilde{x}_2 \\
= & \sum_{j \in \tilde{\mathcal{S}}_2 \cap \mathcal{F}_1} \Big(x^\top(j) G_1(j,j) x(j) - x^\top(j) G_2(j,j) x(j) \Big) \\
& + \sum_{j \in \tilde{\mathcal{S}}_1 \cap \mathcal{F}_2} \Big(x^\top(j) G_2(j,j) x(j) - x^\top(j) G_1(j,j) x(j) \Big) \\
\leq & \; 0
\end{aligned}
\tag{51}
$$

Thus, equilibrium E_1 is stable. $\square$